Inverse Functions

What is an Inverse Function?

An inverse function is a function that will "undo" anything that the original function does. For example, we all have a way of tying our shoes, and how we tie our shoes could be called a function. So, what would be the inverse function of tying our shoes? The inverse function would be "untying" our shoes, because "untying" our shoes will "undo" the original function of tying our shoes.

Let's look at an inverse function from a mathematical point of view. Consider the function $f(x) = 2x - 5$. If we take any value of x and plug it into f(x) what would happen to that value of x? First, the value of x would get multiplied by 2 and then we would subtract 5. The two mathematical operations that are taking place in the function f(x) are multiplication and subtraction. Now let's consider the inverse function. What two mathematical operations would be needed to "undo" f(x)? Division and addition would be needed to "undo" the multiplication and subtraction. A little farther down the page we will find the inverse of $f(x) = 2x - 5$, and hopefully the inverse function will contain both division and addition (see example 5).

Notation

If f(x) represents a function, then the notation $f^{-1}(x)$, read "f inverse of x", will be used to denote the inverse of f(x). Similarly, the notation $g^{-1}(x)$, read "g inverse of x", will be used to denote the inverse of g(x).

Note: $f^{-1}(x) \neq \dfrac{1}{f(x)}$. It is very important not to confuse function notation with negative exponents.

Does the Function have an Inverse?

Not all functions have an inverse, so it is important to determine whether or not a function has an inverse before we try and find the inverse. If a function does not have an inverse, then we need to realize the function does not have an inverse so we do not waste time trying to find something that does not exist.

So how do we know if a function has an inverse? To determine if a function has an inverse function, we need to talk about a special type of function called a **One-to-One Function**. A one-to-one function is a function where each input (x-value) has a unique output (y-value). To put it another way, every time we plug in a value of x we will get a unique value of y, the same y-value will never appear more than once. A one-to-one function is special because only one-to-one functions have an inverse function.

Note: Only One-to-One Functions have an inverse function.

Examples – Now let's look at a few examples to help demonstrate what a one-to-one function is.

Example 1: Determine if the function $f = \{(7, 3), (8, -5), (-2, 11), (-6, 4)\}$ is a one-to-one function.

The function f is a one-to-one function because each of the y-values in the ordered pairs is unique; none of the y-values appear more than once. Since the function f is a one-to-one function, the function f must have an inverse.

Example 2: Determine if the function h = {(–3, 8), (–11, –9), (5, 4), (6, –9)} is a one-to-one function.

The function h is not a one-to-one function because the y-value of –9 is not unique; the y-value of –9 appears more than once. Since the function h is not a one-to-one function, the function h does not have an inverse.

Remember that only one-to-one function have an inverse.

Is the Function a One-to-One Function?

We can determine if functions are one-to-one by looking at ordered pairs and determining if each of the y-values is unique, but what if we do not have ordered pairs? We could create ordered pairs by plugging different x-values into the function and finding the corresponding y-values giving us some ordered pairs. Rather than spending time creating ordered pairs, why not consider looking at the entire graph of the function instead? By looking at the entire graph rather than a few points, we should still be able to determine if the function is a one-to-one function or not.

In looking at the graph of the function we can determine if a function is a one-to-one function or not by applying the Horizontal Line Test, or HLT. If the graph of the function passes the Horizontal Line Test, then the function is a one-to-one function. If the graph of the function fails the Horizontal Line Test, then the function is not a one-to-one function.

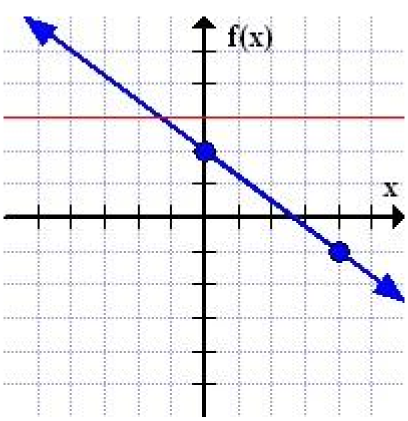
Horizontal Line Test – The HLT says that a function is a one-to-one function if there is no horizontal line that intersects the graph of the function at more than one point.

By applying the Horizontal Line Test not only can we determine if a function is a one-to-one function, but more importantly we can determine if a function has an inverse or not.

Examples – Now let's look at a few examples to help explain the Horizontal Line Test.

Example 3: Determine if the function $f(x) = -\frac{3}{4}x + 2$ is a one-to-one function.

To determine if f(x) is a one-to-one function, we need to look at the graph of f(x). Since f(x) is a linear equation the graph of f(x) is a line with a slope of –3/4 and a y-intercept of (0, 2).

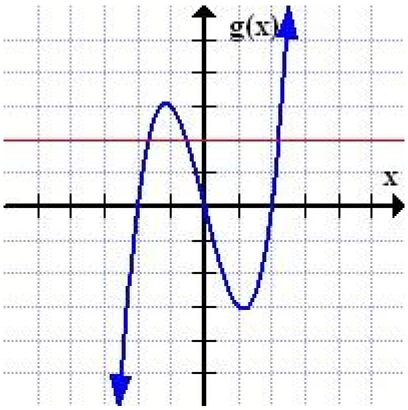


In looking at the graph, you can see that any horizontal line (shown in red) drawn on the graph will intersect the graph of f(x) only once.

Therefore, f(x) is a one-to-one function and f(x) must have an inverse.

Example 4: Determine if the function $g(x) = x^3 - 4x$ is a one-to-one function.

To determine if g(x) is a one-to-one function, we need to look at the graph of g(x).



In looking at the graph, you can see that the horizontal line (shown in red) drawn on the graph intersects the graph of g(x) more than once.

Therefore, g(x) is not a one-to-one function and g(x) does not have an inverse.

How to Find the Inverse Function

Now that we have discussed what an inverse function is, the notation used to represent inverse functions, one-to-one functions, and the Horizontal Line Test, we are ready to try and find an inverse function.

Here are the steps required to find the inverse function:

Step 1: Determine if the function has an inverse. Is the function a one-to-one function? If the function is a one-to-one function, go to step 2. If the function is not a one-to-one function, then say that the function does not have an inverse and stop.

Step 2: Change f(x) to y.

Step 3: Switch x and y.

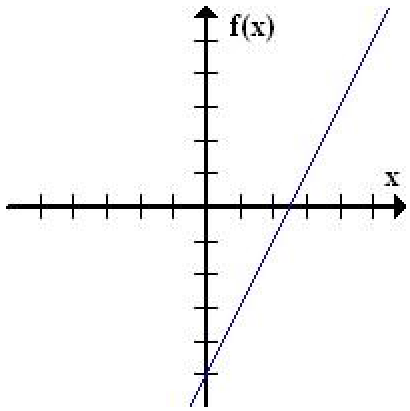
Step 4: Solve for y.

Step 5: Change y back to $f^{-1}(x)$.

By following these 5 steps we can find the inverse function. Make sure that you follow all 5 steps. Many people will skip step 1 and just assume that the function has an inverse; however, not every function has an inverse, because not every function is a one-to-one function. Only functions that pass the Horizontal Line Test are one-to-one functions and only one-to-one functions have an inverse.

Examples – Now let's use the steps shown above to work through some examples of finding inverse functions.

Example 5: If f(x) = 2x – 5, find the inverse.



This function passes the Horizontal Line Test which means it is a one-to-one function that has an inverse.

$y = 2x - 5$ Change f(x) to y.

$x = 2y - 5$ Switch x and y.

$x + 5 = 2y$

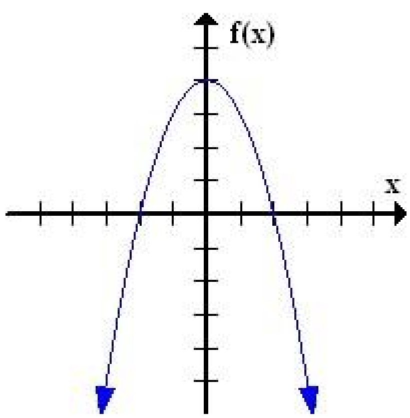
$\frac{x+5}{2} = y$

Solve for y by adding 5 to each side and then dividing each side by 2.

$f^{-1}(x) = \frac{x+5}{2} = \frac{1}{2}x + \frac{5}{2}$ Change y back to $f^{-1}(x)$.

Therefore, $f^{-1}(x) = \frac{x+5}{2}$ or $f^{-1}(x) = \frac{1}{2}x + \frac{5}{2}$.

Example 6: If $f(x) = -x^2 + 4$, find $f^{-1}(x)$.



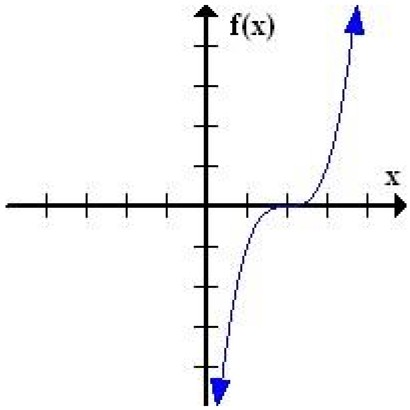
This function does not pass the Horizontal Line Test which means it is not a one-to-one function.

$f^{-1}(x)$ does not exist

f(x) is not a one-to-one, so $f(x) = -x^2 + 4$ does not have an inverse.

Therefore, $f^{-1}(x)$ does not exist.

Example 7: If $f(x) = (x - 2)^3$, find the inverse.



This function passes the Horizontal Line Test which means it is a one-to-one function that has an inverse.

$y = (x - 2)^3$ — Change f(x) to y.

$x = (y - 2)^3$ — Switch x and y.

$\sqrt[3]{x} = \sqrt[3]{(y-2)^3}$

$\sqrt[3]{x} = y - 2$

$\sqrt[3]{x} + 2 = y$

Solve for y by taking the cube root of each side and then adding 2 to each side.

Change y back to $f^{-1}(x)$.

Therefore, $f^{-1}(x) = \sqrt[3]{x} + 2$.

Addition Examples

If you would like to see more examples of finding inverse functions, just click on the link below.

Additional Examples

Practice Problems

Now it is your turn to try a few practice problems on your own. Work on each of the problems below and then click on the link at the end to check your answers.

Problem 1: If $f(x) = \frac{4x-3}{2x+1}$, find $f^{-1}(x)$.

Problem 2: If $f(x) = \frac{5}{6}x - \frac{3}{4}$, find $f^{-1}(x)$.

Problem 3: If $f(x) = -(x + 2)^2 - 1$, find $f^{-1}(x)$.

Problem 4: If $f(x) = -3x + 11$, find $f^{-1}(x)$.

Problem 5: If $f(x) = \sqrt[5]{x+2} - 3$, find $f^{-1}(x)$.

Problem 6: If $f(x) = \frac{2x-5}{3}$, find $f^{-1}(x)$.

Solutions to Practice Problems