Logarithms

Don't Panic!
The objective of this supplement is to build confidence in using Logarithms. Logarithms cause most students some real grief, however the goal here is to show where the Logarithm and its properties come from as well as give examples. Sample problems are provided with solutions at the end.

Reworking exponentials
You have come across exponentials before that look like the following

\[ 3^2 = 9 \quad 5^3 = 125 \quad 2^{10} = 1024 \]

It may seem totally bizarre, but sometimes it is useful to rework this equations by introducing a function called a Logarithm.

\[ \log_3(9) = 2 \quad \log_5(125) = 3 \quad \log_2(1024) = 10 \]

For example, instead of saying 3 squared equals what, we say 3 to what power gives 9? All this can be written very generically as follows.

\[ b^x = y \quad \text{is the same as} \quad \log_b(y) = x \]

To switch between the two follow the steps below (And don't make it too hard).

Convert to logarithms
- The base raised to the power remains as the subscript to the log.
- The number opposite the equal sign gets brought over to go with the log.
- The power gets swapped across the equal sign.

Convert to exponents
- The log is removed, the subscript becomes a regular number.
- The number across the equal sign gets brought across to become the power.
- The number at the end of the log goes across the equal sign.

Sample questions
Switch the following from logarithms to exponents or vice versa. Answers are at the end.

1. \[ 6^2 = 36 \]
2. \[ 7^3 = 2401 \]
3. \[ \log_3(81) = 2 \]
4. \[ \log_2(8) = 3 \]

Note: Instead of pure numbers we can also use variables and everything still works.
Special cases
In general computing logarithms can easily be beyond our scope. However there are some special cases that prove useful.

Case #1
What is the log of zero? To determine this write this as an equation and use our definition.
\[ \log_b(0) = x \quad b^x = 0 \]
Since no number except 0 can be raised to any power to get zero, the log of zero is undefined.

Case #2
What is the log of 1? Follow the same steps.
\[ \log_b(1) = x \quad b^x = 1 \]
A number must be raised to the zero power to be 1, so the log of 1 is always zero.

Case #3
Lastly consider the case
\[ \log_b(b) = x \quad b^x = b \]
Clearly x = 1.

So we have three special cases.
- \( \log_b(0) \) = undefined
- \( \log_b(1) \) = 0
- \( \log_b(b) \) = 1

Sample questions
Solve the following.
1. \( \log_{10}(1) \)
2. \( \log_{1697}(1697) \)
3. \( \log_{x-7}(x-7) \)
4. \( \log_{x^2}(1) \)
5. \( \log_x(0) \)

Multiplying or dividing logs
When multiplying numbers with powers and like bases we can simply add exponents.
\[ 2^3 \cdot 2^5 = 2^{3+5} \]
Likewise when we divide we subtract exponents.
\[ 5^3 \div 5^2 = 5^{3-2} \]
It turns out that logarithms obey these same relationships.
\[ \log_b(x \cdot y) = \log_b(x) + \log_b(y) \]
\[ \log_b(x \div y) = \log_b(x) - \log_b(y) \]

**Examples**
- \( \log_6(2) + \log_6(3) = \log_6(2 \cdot 3) = 1 \)
- \( \log_3(81) - \log_3(3) = \log_3(81 \div 3) = 2 \)

**Sample questions**
Combine the following logarithms together or break them up into two terms.
1. \( \log_5(12) - \log_5(3) \)
2. \( \log_9(2) + \log_9(13) \)
3. \( \log_{10}(17 \cdot 66) \)
4. \( \log_{32}(125 \div 63) \)

Combine the following and express as an exponential. Solve for \( x \).
5. \( \log_4(2) + \log_4(x) = 2 \)
6. \( \log_3(x) + \log_3(9) = 3 \)

**Logarithms with powers**
What if we have a logarithm of a number raised to a power? Check out some examples using our previous results.
- \( \log_4(2^3) = \log_4(2 \cdot 2 \cdot 2) \)
  \[ \log_4(2^3) = \log_4(2) + \log_4(2) + \log_4(2) \]
  \[ \log_4(2^3) = 3 \cdot \log_4(2) \]

- \( \log_7(6^4) = \log_7(6 \cdot 6 \cdot 6 \cdot 6) \)
  \[ \log_7(6^4) = \log_7(6) + \log_7(6) + \log_7(6) + \log_7(6) \]
  \[ \log_7(6^4) = 4 \cdot \log_7(6) \]

In both cases the bottom line is that the power inside the logarithm ended up being brought out as a coefficient.
\[ \log_b(x^p) = p \cdot \log_b(x) \]
**Sample questions**

Express the following without exponents.

1. \( \log_{23}(60^{42}) \)
2. \( \log_{5}(14^4) \)

Simplify the following.

3. \( \log_{3}(3^{22}) \)
4. \( \log_{2}(4) \)

Combine the following logarithms

5. \( 3 \cdot \log_{23}(x) - 4 \cdot \log_{25}(y) \)
6. \( 2 \cdot \log_{9}(z) + 8 \cdot \log_{25}(w) \)

**Logarithms and exponentials**

Logarithms and exponentials act almost as inverse functions, one undoes the other.

\[
\log_b(b^x) = x \quad b^{\log_b x} = x
\]

**Sample questions**

Simplify the following.

1. \( \log_{2} 8 \)
2. \( 3^{\log_{3} 72} \)
3. \( 7^{\log_{7} 10} \)

**Change of Base**

A useful property can be seen by asking what is the division of logarithms with a common base?

\[
Z = \frac{\log_b x}{\log_b y}
\]

To answer this question multiply the denominator over and use the power property of logarithms.

\[
Z \cdot \log_b y = \log_b x
\]

\[
\log_b y^Z = \log_b x
\]

Because the log bases are the same we can say then

\[
y^Z = x
\]

To solve for \( Z \), and hence the original question, use the definition of logarithms.

\[
\log_x x = Z
\]

And so the answer to the original question is.

\[
\log_x x = \frac{\log_b x}{\log_b y}
\]
This is known as the change of base. It is useful if one wants to divide logarithms with like bases or, more commonly, compute the logarithm by changing to a more convenient base.

Sample questions
Divide the following
1. \( \frac{\log_{32}27}{\log_{32}3} \)
2. \( \frac{\log_536}{\log_56} \)

Solve the following by a change of base
3. \( \log_{81}27 \) *Help: change base to 3*
4. \( \log_48 \) *Help: change base to 2*

Special logs
While a logarithm can have almost any base, two special bases get used the most often and are on most scientific calculators. These involve base 10 and base e \( (e \approx 2.71...) \).
The number is an irrational number just like \( \pi \).

- \( \log(x) = \log_{10}(x) \)
- \( \ln(x) = \log_{e}(x) \)

If you need to calculate a logarithm with a base other than 10 or e, you can always use the change of base equation to enter it into your calculator.

Solutions
*Reworking the exponentials*
1. \( \log_6(36) = 2 \)
2. \( \log_7(2401) = 3 \)
3. \( 9^2 = 81 \)
4. \( 2^3 = 8 \)

*Special cases*
1. 0
2. 1
3. 1
4. 0
5. undefined
**Multiplying and Dividing**

1. \( \log_5(4) \)
2. \( \log_9(26) \)
3. \( \log_{99}(17) + \log_{99}(66) \)
4. \( \log_{32}(125) - \log_{32}(63) \)

5. \( 4^2 = 2x \quad x = 8 \)
6. \( 3^3 = 9x \quad x = 3 \)

**Logarithms with powers**

7. \( 42 \cdot \log_{23}(60) \)
8. \( 4 \cdot \log_5(14) \)

9. \( 22 \cdot \log_3(3) = 22 \)
10. \( \log_2(4) = \log_2(2^2) \quad \log_2(4) = 2 \cdot \log_2(2) = 2 \)

11. \( \log_{25}\left(\frac{x^3}{y^4}\right) \)
12. \( \log_9(z^2 \cdot w^8) \)

**Logarithms and exponentials**

1. \( \log_2 8 = \log_2 2^3 = 3 \)
2. \( 3^{\log_3 72} = 72 \)
3. \( 7^{\log_7 10} = 10 \)

**Change of Base**

1. \( \frac{\log_{32} 27}{\log_{32} 3} = \log_3 27 = 3 \)
2. \( \frac{\log_7 36}{\log_7 6} = \log_6 36 = 2 \)

3. \( \log_8 27 = \frac{\log_3 27}{\log_3 81} = \frac{3}{4} \)
4. \( \log_4 8 = \frac{\log_2 8}{\log_2 4} = \frac{3}{2} \)
### Summary of Properties

**Definition**

\[ b^x = y \quad \log_b(y) = x \]

**Special logarithms**

\[ \log_b(0) = \text{undefined} \quad \log_b(1) = 0 \quad \log_b(b) = 1 \]

\[ \log(x) = \log_{10}(x) \quad \ln(x) = \log_e(x) \]

**Addition or subtraction of logarithms**

\[ \log_b(x \cdot y) = \log_b(x) + \log_b(y) \quad \log_b(x \div y) = \log_b(x) - \log_b(y) \]

**Logarithms including powers**

\[ \log_b(x^p) = p \cdot \log_b(x) \]

**Logarithms and exponents**

\[ \log_b(b^x) = x \quad b^{\log_b(x)} = x \]

**Change of base**

\[ \log_y(x) = \frac{\log_b(x)}{\log_b(y)} \]

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