Background
The original research objective behind this piece was to develop a totally general expression for the scalar and vector potentials describing electrodynamics. The result was a surprising understanding behind the meaning of Maxwell's equations. This piece will trace the steps leading up to the discovery and explore its meaning.

The Physics
The physics at the heart of the matter is Maxwell's four famous equations.

\[ \nabla \cdot \mathbf{B} = 0 \quad (1a) \]

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1b) \]

\[ \nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} \quad (1c) \]

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1d) \]

Equations (1a) and (1b) can be satisfied generally by use of a scalar potential \( V \) and a vector potential \( A \).

\[ \mathbf{B} = \nabla \times A \quad \mathbf{E} = \nabla V - \frac{\partial A}{\partial t} \quad (2) \]

The bulk of the work focuses on cleaning up what becomes of the equations that remain upon substitution of (2) into (1).

\[ \nabla \cdot \left( \nabla V - \frac{\partial A}{\partial t} \right) = \frac{\rho}{\varepsilon_0} \quad (3a) \]

\[ \nabla \times \left( \nabla \times A \right) = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left( \nabla V - \frac{\partial A}{\partial t} \right) \quad (3b) \]

Reducing the math
The equations (3a) and (3b) need first to be expanded using a summation over components. Equation (3a) becomes

\[ \sum_i \left[ \frac{\partial^2 V}{\partial x_i^2} - \frac{\partial^2 A_i}{\partial x_i \partial t} \right] = \frac{\rho}{\varepsilon_0} \quad (4) \]

Equation (3b) involves a double cross product. It can be shown using properties of cross products that this becomes.

\[ \sum_i \left[ \frac{\partial^2 A_i}{\partial x_i \partial x_i} - \frac{\partial^2 A_n}{\partial x_i^2} \right] = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \frac{\partial}{\partial t} \left[ \frac{\partial^2 V}{\partial x_n \partial t} - \frac{\partial^2 A_n}{\partial t^2} \right] \quad (5) \]
The potentials in (2) must have a physical source or sources scattered throughout space and time. We can therefore express our potentials as an integral over all space-time with some function \( G \) specifying the effect of sources scattered through space and time.

\[
V(\mathcal{L}) = \int_{\text{all space-time}} G_V(\mathcal{L}, r_0) \, d^4 r_0 \quad \text{and} \quad A_n(\mathcal{L}) = \int_{\text{all space-time}} G_{An}(\mathcal{L}, r_0) \, d^4 r_0
\]

The convention will be maintained that vectors containing time and space will be underlined while vectors containing just three spacial coordinates will have an arrow.

It will be noted that there is no inherent enforcement of an advanced or retarding potential. The reason is twofold. First, the mathematics becomes simpler by not enforcing these constraints. Secondly it is desired to make the conclusions as general as possible, even to include the possibility of non-local behavior.

So after substituting (6) into (4) and (5) one can obtain, after eliminating the integration.

\[
\sum_l \left[ \frac{\partial^2 G_V}{\partial x_i^2} - \frac{\partial^2 G_{Al}}{\partial x_i^2} \right] \frac{\rho(\mathcal{L}_0)}{\varepsilon_0} \cdot \delta(\mathcal{L}, \mathcal{L}_0) = \sum_l \left[ \frac{\partial^2 G_{Al}}{\partial x_n x_l} - \frac{\partial^2 G_{An}}{\partial x_l^2} \right] \mu_0 J_n(\mathcal{L}) \cdot \delta(\mathcal{L}, r_0) + \mu_0 \varepsilon_0 \left[ \frac{\partial^2 G_V}{\partial x_n t} - \frac{\partial^2 G_{An}}{\partial t^2} \right] \tag{7}
\]

The arguments of \( G_V \) and \( G_{An} \) are suppressed for convenience.

What (7) and (8) present is a nasty system of four partial differential equations. To attempt to simplify the equations apply the Fourier transform and reduce the equations down to a system of algebraic equations. Let \( F \) represent the Fourier transform of the different functions.

\[
\sum_l \left[ k_n k_l F_{Al} - k_l^2 F_{V} \right] e^{-i(\vec{k} \cdot \vec{r}_0 + \omega t)} = \frac{\rho(\mathcal{L}_0)}{\varepsilon_0 (2\pi)^2} e^{-i(\vec{k} \cdot \vec{r}_0 + \omega t)} \tag{9}
\]

\[
\sum_l \left[ k_l^2 F_{An} - k_n k_l F_{Al} \right] e^{-i(\vec{k} \cdot \vec{r}_0 + \omega t)} = \frac{\mu_0}{(2\pi)^2} J_n(\mathcal{L}_0) e^{-i(\vec{k} \cdot \vec{r}_0 + \omega t)} + \mu_0 \varepsilon_0 \left[ \omega^2 F_{An} - k_n \omega F_{V} \right] \tag{10}
\]

Now it remains to solve this linear system of equations.

**The solution**

At this point the majority of the work proceeded well until the final stage when \( F \) was to be solved for. At this point all the Fourier transformed function dropped out leaving the following.

\[
0 = \left[ \omega \cdot \rho + k_x \cdot J_x + k_y \cdot J_y + k_z \cdot J_z \right] e^{-i(\vec{k} \cdot \vec{r}_0 + \omega t)} \tag{11}
\]

The intermediate steps are highlighted in the Appendix below. Equation (11) is an interesting result as it is a boundary condition on the charges creating the fields rather
than the fields themselves. This suggests that Maxwell's equations rely on some aspect of the behavior of matter. It can be seen that (11) can be transformed into.

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z}$$

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J}$$  \hspace{1cm} (12)

This however is local charge conservation, in other words charge conservation where the movement of charges is limited to moving from a place to another place via some continuous path. Therefore because the substitutions to (2) are general for all cases, Maxwell's equations must rest on charge conservation even though it does not explicitly state it in the equations.

Another way to look at it can be seen by using (3a) and (3b) to solve for the time derivative of charge density and the divergence of the current density.

$$\dot{\rho} = \varepsilon_0 \left( \nabla^2 \dot{V} - \vec{\nabla} \cdot \vec{\nabla} \cdot \vec{A} \right)$$ \hspace{1cm} (13a)

$$\vec{\nabla} \cdot \vec{J} = \varepsilon_0 \left( \vec{\nabla} \cdot \vec{\nabla} \cdot \vec{A} - \nabla^2 \dot{V} \right)$$ \hspace{1cm} (13b)

Putting (13a) and (13b) into (12) gives

$$0 = \varepsilon_0 \left( \nabla^2 \dot{V} - \vec{\nabla} \cdot \vec{\nabla} \cdot \vec{A} + \vec{\nabla} \cdot \vec{\nabla} \cdot \vec{A} - \nabla^2 \dot{V} \right)$$

$$0 = 0$$

Any solution to (1) must naturally be consistent with local charge conservation (12).

**Relating the scalar and vector potentials**

The solutions to (12) cannot help but satisfy local charge conservation. However can nothing more be said about this potentials? Equation (4) is a partial differential equation that can be worked on by applying the Fourier transform.

$$\omega \vec{k} \cdot \vec{A}_k - k^2 \vec{F}_v = \frac{\vec{F}_v}{\varepsilon_0}$$  \hspace{1cm} (14)

Again, let the F denote the Fourier transforms of the particular functions. $F_v$ is the easiest quantity to solve for.

$$F_v = -\frac{\vec{F}_v}{\varepsilon_0 k^2} + \frac{\omega}{k^2} \vec{k} \cdot \vec{F}_A$$ \hspace{1cm} (15)

$V$ can be solved for by performing the inverse transform. It must first be remembered that there remain two other functions that are transforms. These should also be re-expressed as their original functions.
\[ V(\mathbf{r}) = \frac{-1}{(2\pi)^2 \epsilon_0} \int \frac{1}{k^2} e^{i(k\cdot r + \omega t)} d^3 k d\omega \left[ \int \frac{\rho(r')}{(2\pi)^2} e^{-i(k\cdot r' + \omega t')} d^3 r' d t' \right] + \frac{1}{(2\pi)^2} \int \frac{\omega}{k^2} e^{i(k\cdot \mathbf{r} + \omega t)} d^3 k d\omega \left[ \int \frac{\tilde{A}(\mathbf{r}')}{(2\pi)^2} e^{-i(k\cdot r' + \omega t')} d^3 r' d t' \right] \]

(16)

All these transforms are over space and time. (16) Can be reworked to isolate the space and time variables of integration.

\[ V(\mathbf{r}) = \int \rho(\mathbf{r}') d^3 r' d t' \left[ -\frac{1}{(2\pi)^2 \epsilon_0} \int \frac{1}{k^2} e^{i(k\cdot r' + \omega t')} e^{i\omega(t-t')} d^3 k d\omega \right] + \int \tilde{A}(\mathbf{r}') d^3 r' d t' \left[ \frac{1}{(2\pi)^2 \epsilon_0} \int \frac{\omega}{k^2} e^{i(k\cdot \mathbf{r}' + \omega t')} e^{i\omega(t-t')} d^3 k d\omega \right] \]

(17)

It can be shown that the bracketed portions of this integration can be solved to reduce this integral as follows.

\[ V(\mathbf{r}) = \int \rho(\mathbf{r}') d^3 r' d t' \left[ -\frac{\delta(t-t')}{4\pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|} \right] + \int \tilde{A}(\mathbf{r}') d^3 r' d t' \left[ -\frac{\mathbf{r} - \mathbf{r}'}{4\pi \epsilon_0 |\mathbf{r} - \mathbf{r}'|^3} \frac{\partial}{\partial t} \delta(t-t') \right] \]

(18)

\[ V(\mathbf{r}) = -\frac{1}{4\pi \epsilon_0} \int d^3 r' \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi} \frac{\partial}{\partial t} \int d^3 r' \frac{\tilde{A}(\mathbf{r}', t) \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \]

(19)

A couple of aspects of (19) are interesting.

- \( V(\mathbf{r}) \) involves integrals which are not delayed in time. In other words they are instantaneous.

- The second integrand reflects the potential of an electric dipole. So a changing \( \mathbf{A} \) acts like a dipole, creating a \( V \).

Although the results are not as pretty, one can follow a similar course with \( \mathbf{A} \) by taking the partial derivative of (4) with respect to time and employing (12). This gives

\[ \sum_l \left[ \frac{\partial^2 V}{\partial x_l^2 t} - \frac{\partial^2 A_l}{\partial x_l t} \right] = -\frac{1}{\epsilon_0} \sum_l \frac{\partial J_l}{\partial x_l} \]

(20)

\[ \frac{\partial^2 V}{\partial x_l t} - \frac{\partial^2 A_l}{\partial t^2} = -\frac{J_l}{\epsilon_0} + R_l \]

(21)

\( \mathbf{R} \) is a vector with a divergence of zero. Following a now-familiar theme, take the Fourier transform of (21).

\[ -\omega \cdot k \cdot F_V + \omega^2 \cdot F_A = -\frac{F_R}{\epsilon_0} + F_R \]

(22)
We can in fact solve for the components of $F_A$.

$$F_{Al} = \frac{-F_{Jl}}{\varepsilon_0 \omega^2} + \frac{k_l}{\omega^2} \cdot F_V$$

We can perform the inverse transform similar as that to obtain (18) to solve for $A$.

$$\tilde{A}(r) = \frac{1}{(2\pi)^2 \varepsilon_0} \int dt_0 \tilde{R}(\tilde{r}, t_0) - \tilde{J}(\tilde{r}, t_0) \cdot |t - t_0| + \frac{1}{(2\pi)^2} \tilde{\nabla} \int dt_0 V(\tilde{r}, t_0)$$

(24)

It is of note that neither (19) nor (24) adhere to the orthodox tenet of causality. Equation (19) has an integration over all space at the instant in question and (24) involves integration over all time.

Equation (24) is limited in its value because of the unknown $R$. However we can circumvent this problem by taking the Fourier transform of (1d) and substituting (23) into it.

$$\sum_l \left\{ k_n k_l \left[ \frac{F_{Rl}}{\omega} + \frac{k_l}{\omega} \cdot F_V - \frac{F_{Jl}}{\varepsilon_0 \omega^2} \right] - k_i^2 \left[ \frac{F_{Rn}}{\omega^2} + \frac{k_n}{\varepsilon_0 \omega^2} \cdot F_V F_{Jn} - \frac{F_{Jn}}{\varepsilon \omega^2} \right] \right\} = -\mu_0 \cdot F_{Jn}$$

$$+ \mu_0 \varepsilon_0 \left\{ k_i \left[ \frac{F_{Rn}}{\omega} + \frac{k_n F_V}{\varepsilon_0 \omega^2} - \frac{F_{Jn}}{\varepsilon_0 \omega^2} \right] \right\}$$

(25)

Rearranging (25) and remembering that $R$ has zero divergence gives.

$$\frac{1}{c^2} \sum_l k_i^2 \left[ F_{Rn} - \frac{k_n \sum_l k_i^2}{\omega} - \frac{k_l \sum_l k_i^2}{\omega} \right] F_V - \frac{\sum_l k_i^2}{\varepsilon_0 \omega} F_{Jn} - \frac{k_n}{\varepsilon \omega} F_J = -\frac{\omega^2}{c^2} \sum_l k_i^2 F_{Al}$$

(26)

Equation (23) can be arranged to accept the substitution of (26) into it. This produces

$$-\left[ \frac{\omega^2}{c^2} - \sum_l k_i^2 \right] F_{Al} = - \left( \frac{k_l}{\varepsilon_0 \omega} \right) F_V + \mu_0 \cdot F_{Jl} + \left[ \frac{\omega k_i}{c^2} - \frac{k_i \sum_l k_i^2}{\omega} \right] F_V$$

(27)

Performing the inverse transform gives.

$$\square A(\tilde{r}) = \mu_0 \tilde{\nabla} - \tilde{\nabla} \left( \frac{\partial V}{\partial t} \right) - \tilde{\nabla} \int_{-\infty}^t \rho(\tilde{r}, t_0) \cdot \tilde{\nabla}^2 V(\tilde{r}, t_0) \cdot dt_0$$

(28)

Imposing (1d) creates a relationship for $A$ that looks consistent with orthodox causality.

**The roles of $V$ and $A$**

An interesting asymmetry between $V$ and $A$ as seen in (19) and (24). The scalar potential is generated instantaneously by sources throughout space. On the other hand (24) shows that $A$ is the cumulative effect of sources at the same place but over time.
Conclusion
Maxwell's equations form a symbiotic relationship with charge conservation. The Maxwell's equations depend on charge conservation and, like a dutiful steward, guarantees charge conservation. In fact Maxwell's equations can be rewritten as a set of three equations.

\[ 0 = \frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} \]  
(29a)

\[ \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \]  
(29b)

\[ \nabla \cdot \vec{B} = 0 \]  
(29c)

By taking the divergence of (29b) one gets that the divergence of \( \vec{B} \) must be independent of time. Therefore (29c) is simply a boundary condition on the other two. Moreover (19), which assumed no Gauge, shows that \( V \) and \( A \) are simultaneously affected by \( \rho \).

Taken as a package, (19) and (28) do not strictly adhere to a local interpretation of physical processes. However the sources of the fields as well as the charges must be conserved in a local fashion. Thus electromagnetic fields and matter appear to obey a hybridization of local and non-local physics.

Appendix: Solving the system
Equations (9) and (10) can be expanded out as a system of four equations and rewritten as the following.

\[ \omega k_x \cdot F_{Ax} + \omega k_y \cdot F_{Ay} + \omega k_z \cdot F_{Az} = k^2 \cdot F_y = \frac{\rho \left( r_0 \right)}{\epsilon_0} \cdot e^{-i \left( k \cdot \vec{r}_0 + \omega t \right)} \]  
(i)

\[ \left( k^2 - k_x^2 - \frac{\omega^2}{c^2} \right) \cdot F_{Ax} - k_x k_y \cdot F_{Ay} - k_x k_z \cdot F_{Az} + \frac{k_y \omega}{c^2} \cdot F_y = \mu_0 J_x \left( r_0 \right) \cdot e^{-i \left( k \cdot \vec{r}_0 + \omega t \right)} \]  
(ii)

\[ -k_x k_y \cdot F_{Ax} + \left( k^2 - k_y^2 - \frac{\omega^2}{c^2} \right) \cdot F_{Ay} - k_x k_z \cdot F_{Az} + \frac{k_z \omega}{c^2} \cdot F_y = \mu_0 J_y \left( r_0 \right) \cdot e^{-i \left( k \cdot \vec{r}_0 + \omega t \right)} \]  
(iii)

\[ -k_x k_z \cdot F_{Ax} - k_x k_y \cdot F_{Ay} + \left( k^2 - k_z^2 - \frac{\omega^2}{c^2} \right) \cdot F_{Az} + \frac{k_z \omega}{c^2} \cdot F_y = \mu_0 J_z \left( r_0 \right) \cdot e^{-i \left( k \cdot \vec{r}_0 + \omega t \right)} \]  
(iv)

With \( k^2 \) being the magnitude squared of \( k \).

Equation (i) can be used to solve for \( F_{Ax} \) in terms of the other functions.

6.
\[ F_{Ax} = -\frac{k_y}{k_x} F_y - \frac{k_z}{k_x} F_z + \frac{k^2}{\omega k_x} F_y + \frac{\rho(r_0)}{\omega k_x \epsilon_0} e^{-i(k \cdot \hat{r}_0 + \omega t)} \] (v)

We can substitute (v) into equation (ii) and simplify.

\[ k_y \left( \omega^2 - c^2 k_x^2 \right) F_{Ay} + \frac{k^2}{c^2 k_x} F_y + \frac{c^2 k^4 + k_x^2 \omega^2 - k_x^2 c^2 - \omega^2 k_x^2}{c^2 k_x \omega} F_y = \mu_0 J_x(r_0) e^{-i(k \cdot \hat{r}_0 + \omega t)} + \frac{\omega^2 + c^2 k_x^2 - c^2}{c^2 k_x \omega} \frac{\rho(r_0)}{\epsilon_0} e^{-i(k \cdot \hat{r}_0 + \omega t)} \] (vi)

Likewise (v) can be substituted into (iii) and (iv). Only in these cases it is convenient to solve for two of the unknown functions.

\[ F_{Ay} = \frac{k_y}{\omega} F_y + \frac{k^2}{c^2 k_x^2 - \omega^2} \mu_0 J_y(r_0) e^{-i(k \cdot \hat{r}_0 + \omega t)} + \frac{k_y c^2}{\omega (c^2 k_x^2 - \omega^2)} \frac{\rho(r_0)}{\epsilon_0} e^{-i(k \cdot \hat{r}_0 + \omega t)} \] (vii)

\[ F_{Az} = \frac{k_z}{\omega} F_y + \frac{k^2}{c^2 k_x^2 - \omega^2} \mu_0 J_z(r_0) e^{-i(k \cdot \hat{r}_0 + \omega t)} + \frac{k_z c^2}{\omega (c^2 k_x^2 - \omega^2)} \frac{\rho(r_0)}{\epsilon_0} e^{-i(k \cdot \hat{r}_0 + \omega t)} \] (viii)

We can now put both (vii) and (viii) to try to solve for \( F_y \). However when one does this \( F_y \) drops out and one is left with

\[ 0 = \omega \left( \frac{\rho(r_0)}{\epsilon_0} + \mu_0 c^2 \left[ k_x J_x(r_0) + k_y J_y(r_0) + k_z J_z(r_0) \right] \right) e^{-i(k \cdot \hat{r}_0 + \omega t)} \] (ix)

One solution to (ix) is when \( \omega = 0 \), or a static case. The other possibility is

\[ 0 = \omega \cdot \rho(r_0) + k_x J_x(r_0) + k_y J_y(r_0) + k_z J_z(r_0) \] (x)

This is the result given as equation (11) above.

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