The Klein-Gordon equation and Antimatter
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Introduction
Physicists employ different equations to solve for the quantum behavior of matter depending on the nature of the particle. For particles such as electrons and protons the Dirac equation\(^1\) is employed.

\[
i \hbar \frac{\partial \Psi}{\partial t} = \frac{\hbar}{i} \sum_n \alpha_n \left( \frac{\partial \Psi}{\partial x_n} \right) + mc^2 \mathbf{\beta} \cdot \Psi
\] (1)

For photons the Quantum Mechanical equivalent of Maxwell's equation\(^2\) is used

\[
\frac{i \partial \tilde{\Psi}}{c \partial t} = \vec{\nabla} \times \tilde{\Psi}
\] (2)

And for particles without spin the Klein-Gordon equation\(^3,4\) is used. This can be expressed as follows

\[
\nabla^2 \Psi - \frac{1}{c^2} \partial_t^2 \Psi = - \left( \frac{E_0}{c \hbar} \right)^2 \Psi
\] (3)

\(E_0\) is the rest energy and \(\partial_t\) is defined as the partial derivative with respect to time \(t\).

Equations (1) through (3) address matter which, classically, behave the same there should exist a common equation describing all cases. The simplest case is the Klein-Gordon equation and so this will be developed and used.

The objections to the Klein-Gordon equation are three-fold

1. For an initial wave function \(\Psi(r,t=0)\), one must also have some extra initial condition to know how \(\Psi(r,t)\) develops over time. It is desired to have an equation where all one needs to know is \(\Psi(r,t=0)\) to predict all \(\Psi(r,t)\).
2. The \(\Psi(r,t)\) obtained cannot be a probability. It is argued that if the solution is normalized at a given time it must remain normalized.
3. The solutions to do not involve spin and so it can only be useful for spin zero particles.

These objections will be addressed one at a time. Some of the conclusions are unique to this paper, while others echo previous research. In the case of the latter a citation will be included.

Spin
The first objection to the Klein-Gordon is that there is no room for the spin states. However it has been shown not only that the elements of the Dirac equation solutions do not represent states, but that the meaning of spin states is vague\(^5\). Therefore not having

1.
spin poses no fundamental loss. Moreover if one wished to, one could always write $\Psi$ as a column matrix that allows room for spin states and is still compatible with (3).

The need for only one initial condition

The general solution can be expressed as follows

$$\Psi(\vec{r}, t) = \int a(\vec{p}) \cdot e^{\frac{i}{\hbar} (\vec{p} \cdot \vec{r} - E(\vec{p}) \cdot t)} \cdot d^3 p + \int b(\vec{p}) \cdot e^{\frac{i}{\hbar} (\vec{p} \cdot \vec{r} + E(\vec{p}) \cdot t)} \cdot d^3 p$$  \hspace{1cm} (4)

With

$$E(\vec{p}) = \sqrt{c^2 \cdot |\vec{p}|^2 + (E_0)^2}$$  \hspace{1cm} (5)

The first integration in (4) can be interpreted as belonging to states of positive energy $E$ and the second states of negative $E$. Moreover the vector $\vec{p}$ can represent particle momentum for each state.

The generalized solution one can obtain a solution at time $t = 0$.

$$\Psi(\vec{r}, t=0) = \int [a(\vec{p}) + b(\vec{p})] \cdot e^{\frac{i}{\hbar} |\vec{p}| \cdot \vec{r}} \cdot d^3 p$$  \hspace{1cm} (6)

The sum of $a(p)$ and $b(p)$ must therefore be the Fourier transform of the initial wave function with $k = p/\hbar$. As it stands there must be an additional condition to solve for both functions. One way to bypass this objection can is to impose the restriction that a particle can have positive or negative energy, but cannot be a mixture of both. Under this assumption the nonzero function $a(p)$ or $b(p)$ would be the same as the Fourier transform of the initial wave function. Later on we will return to the question of superposition of positive and negative energy particles.

The Probability normalization

If one integrates the modulus squared of the general solution to the Klein Gordon equation above over all space one gets

$$\int_{all\ space} |\Psi(\vec{r}, t)|^2 \cdot d^3 r = h \cdot \int (|a(\vec{p})|^2 + |b(\vec{p})|^2) \cdot d^3 p + 2h \cdot Re \left[ \int a(\vec{p}) \cdot b(\vec{p}) \cdot e^{\frac{2iE}{\hbar} t} \cdot d^3 p \right]$$  \hspace{1cm} (7)

The line over the function $b(p)$ signifies the complex conjugate. It will be noted that under the above restriction the time dependent portion vanishes and the integral becomes constant, thus remaining normalizable.

Particles of negative energy

The general solution (4) involves solutions that can be interpreted to be positive and negative energy energy states. So how does a particle of negative energy differ from a particle of positive energy? Comparing the general solution to the Klein-Gordon equation for the case of positive vs. negative energy one is struck by one major difference, namely time is inverted ($t \rightarrow -t$). This can be interpreted to mean that in the
relativistic limit the negative energy particle behaves the same as a particle with positive energy that is moving backwards in time. It is desirable to express these negative energy particles in terms of positive energy particles so we are on familiar ground, so the distinction between how negative and positive energy particles are modeled will then be made as follows

*Particles of positive energy move “forward” in time while particles of negative energy move backwards, in the relativistic limit.*

Additionally for the case of a negative energy particle the momentum and velocity are anti-parallel in the relativistic limit because of the relation involving velocity, relativistic momentum and energy

\[ \vec{u} = c^2 \frac{\vec{P}}{E} \]  

(8)

To further examine the matter in depth, consider a positive energy electron giving off a photon and producing a negative energy electron (represented by \( e^- \)) as shown below in figure 1.

![fig. 1](image)

The downward direction of \( e^- \) depicts its traveling backwards in time. Describing this process from a purely chronological standpoint means it can be written as follows.

\[ e^- + e^- \rightarrow \text{photon} \]

Problems appear to arise from this depiction. The largest objection is that the combined charge prior to the collision does not equate to the final charge (being zero). Moreover it is not clear that the combined energy of a positive and negative energy electron equals the photon energy.

To correct for these problems, imagine replacing the \( e^- \) with a positive energy pseudo-particle that has a positive charge (to conserve charge) called \( e^+ \).

\[ e^- + e^+ \rightarrow \text{photon} \]

Because of this reaction, this pseudo-particle is called an “anti-particle.” It is actually a
fruit of considering the reaction in terms of what goes in vs. what comes out as opposed to ordering the reaction chronologically.

Although the anti-particle needs to have an opposite charge the make the above scenario make sense, it can be explained in the following manner. Suppose an $e^{-}$ is at rest near a large positive charge as shown in figure 2.

![fig. 2](image)

Because of the attractive force the momentum of $e^{-}$ must be towards the positive charge, however because the electron has a negative energy the velocity $u$ must be anti-parallel to momentum. It will be repelled. But suppose the particle is now depicted as a positive energy particle? Now momentum and velocity must be in the same direction, both away from the positive charge (the electron must still be repelled). The only way to make a consistent picture is to have the anti-particle have an opposite charge.

There is one correction that must be made so the above is consistent with experiment. Experimentally a particle/anti-particle pair are combined to produce not one but two photons. This adds symmetry to the explanation above as one can view the positive energy particle becomes negative by collision with a single negative energy photon as shown below in figure 3.

![fig. 3](image)
In summary:

- Negative energy particles can be viewed as positive energy particles moving backwards in time.
- The momentum and velocity of a negative particles are anti-parallel.
- A negative energy particle can be recast as a positive energy anti-particle with the following changes:
  1. The particle moves forward in time.
  2. The momentum 4-vector reverses direction
  3. Charge reverses sign

### Superposition of positive and negative energy states

It was assumed above that a particle can have either a positive or negative energy but cannot be a superposition of the two. But what if there were such a superposition of states for some particle X?

\[ \Psi_{X_{\text{mix}}} = \alpha \Psi_X + \beta \Psi_{X^*} \]

Now the state is a mixture of states moving forward and backwards in time, the resulting interference will vary the probability that the particle be found somewhere in space. One way to justify this can be seen after rewriting the negative energy state as the anti-particle state.

\[ \Psi_{X_{\text{mix}}} = \alpha \Psi_X + \beta \Psi_{\text{anti-X}} \]

Since this state is a mixture of matter and anti-matter the reaction

\[ X + \text{anti-X} \leftrightarrow \text{photon} \]

can occur. With this reaction a possibility the probability of finding the particle somewhere would naturally vary over time.

Yet there is still something more fundamental. In a superposition of states the probability of finding a particle near a given point differs from the classical model. Where constructive interference takes place the probability is higher. Conversely destructive interference reduces the probability. Yet what of time? Since relativity beings time and space together in a similar footing, one would expect a similar process for time. In fact one should redefine the probability density of quantum mechanics to be the probability of finding a particle near a given point and near the time being measured at.

### Conclusion

The Klein-Gordon equation is a viable quantum mechanical equation with relativistic considerations. The result is not only a description of the behavior of a negative energy
particle, but a explanation for what anti-matter is.

Works Cited
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