Law of Sines

Right Triangles

Up to this point all of the triangles we have used in trigonometry have always been right triangles. What if we are given an oblique triangle (a triangle without a right angle) and we are asked to find the missing sides or angles? Simple, turn the oblique triangle into a right triangle by drawing a line in the triangle. Let’s consider the following example.

Example 1: Given \( \triangle ABC \) with \( \angle A = 47^\circ \), \( \angle B = 68^\circ \) and \( a = 18 \). Find side \( b \).

Draw and label the triangle.

Find the value of \( h \).

\[
\sin 64^\circ = \frac{h}{18}
\]

\[
h = 18 \sin 64^\circ
\]

\[
h = 16.18
\]

Find the value of \( b \).

\[
\sin 47^\circ = \frac{16.18}{b}
\]

\[
bsin 47^\circ = 16.18
\]

\[
b = \frac{16.18}{\sin 47^\circ}
\]

\[
b = 22.12
\]

The length of side \( b \) is 22.12.

In finding the value of side \( b \), the two steps used both required the use of sine and \( h \). Let’s look at the same problem again without using any numbers to see if there is some sort of short cut we can take advantage of for future use.

Example 2: Given \( \triangle ABC \) with \( \angle A = A^\circ \), \( \angle B = B^\circ \) and \( a = a \). Find side \( b \).

Draw and label the triangle.

Create two right triangles by drawing the line segment shown in blue and label that line segment as \( h \).
Find the value of h.

\[
\sin B = \frac{h}{a} \quad \text{Use sine to find the value of h by cross multiplying.}
\]

\[
h = a \sin B
\]

Find the value of b.

\[
\sin A = \frac{h}{b} \quad \text{Use sine to find the value of h by cross multiplying.}
\]

\[
h = b \sin A
\]

Both equations are equal to h, so they must be equal to each other.

\[
h = a \sin B \quad \text{a \sin B = b \sin A} \quad \frac{\sin A}{a} = \frac{\sin B}{b}
\]

The last equation gives us a way to find the value of b without needing to find the value of h. This means we can solve oblique triangles without needing to create right triangles.

**Law of Sines**

The equation from above is known as the Law of Sines. The Law of Sines gives a relationship between the sines of angles and the sides of a triangle. The ratio of the sine of an angle and the length of the side opposite the angle is the same for each angle of the triangle. Here is the Law if Sines.

![Law of Sines](image)

The Law of Sines applies to any triangle, even right triangles. If we have a right triangle, we do not need to use the Law of Sines, we should use the basic trigonometry functions we have always used. The key to using the Law of Sines is we must know the value of one angle and the side opposite that angle, we must know angle A and side a, angle B and side b, or angle C and side c. Let’s try a few examples.

**Example 3:** Given \(\triangle ABC\) with \(\angle B = 34^\circ\), \(\angle C = 64^\circ\) and \(a = 5.3\). Find \(b\), \(c\), and \(\angle A\).

Draw and label the triangle.

Determine if the Law of Sines applies. We can use the Law of Sines, if we find \(\angle A\).

\[\angle A = 180^\circ - 64^\circ - 34^\circ = 82^\circ.\]
Find b.
\[
\frac{\sin 82}{5.3} = \frac{\sin 34}{b}
\]
\[b \sin 82 = 5.3 \sin 34\]
\[b = \frac{5.3 \sin 34}{\sin 82}\]
\[b \approx 3.0\]

Find c.
\[
\frac{\sin 82}{5.3} = \frac{\sin 64}{c}
\]
\[c \sin 82 = 5.3 \sin 64\]
\[c = \frac{5.3 \sin 64}{\sin 82}\]
\[c \approx 4.8\]

Therefore, \(\angle A = 82^\circ\), \(b = 3.0\), and \(c = 4.8\).

When finding the missing parts of the triangle you should always try to use the known values to find the missing values. In the example above, we could have used \(b\) to find the value of \(c\), but what if \(b\) is incorrect? To avoid such issues, you should use known information to find the unknowns.

**Example 4:** Given \(\triangle ABC\) with \(\angle A = 28^\circ\), \(\angle B = 66^\circ\) and \(c = 18.2\). Find \(a\), \(b\), and \(\angle C\).

Draw and label the triangle.

Determine if the Law of Sines applies. We can use the Law of Sines, if we find \(\angle C\).

\[\angle C = 180^\circ - 66^\circ - 28^\circ = 86^\circ\]

Find a.
\[
\frac{\sin 86}{18.2} = \frac{\sin 28}{a}
\]
\[a \sin 86 = 18.2 \sin 28\]
\[a = \frac{18.2 \sin 28}{\sin 86}\]
\[a \approx 8.6\]
Find b.

\[
\frac{\sin 86}{18.2} = \frac{\sin 66}{b} \quad \text{Solve by cross multiplying.}
\]

\[b \sin 86 = 18.2 \sin 66\]

\[b = \frac{18.2 \sin 66}{\sin 86}\]

\[b \approx 16.7\]

Therefore, \(\angle C = 86^\circ\), \(a = 8.6\), and \(b = 16.7\).

**Tip:** After finding the missing values, take a second to verify if your answers are possible. To determine if your answers are possible make sure the largest side is opposite the largest angle and the smallest angle is opposite the smallest side. This does not guarantee the answers are correct, but it is a good sign.

**Does the Law of Sines Always Work?**

In the example we have done so far, we have always known two of the angles and one of the sides. In cases where we know two angles and one side, the Law of Sines will always work. What if we know one angle and two sides, one of which is opposite the known angle? To help answer this question, let’s look at the following examples.

**Example 5:** Given \(\triangle ABC\) with \(\angle C = 125^\circ\), \(b = 16.7\) and \(c = 24.6\). Find \(\angle A\), \(\angle B\), and \(\angle C\).

Draw and label the triangle.

**Determine if the Law of Sines applies.** We can use the Law of Sines, to find \(\angle B\).

Find \(\angle B\).

\[
\frac{\sin 125}{24.6} = \frac{\sin B}{16.7} \quad \text{Solve by cross multiplying.}
\]

\[24.6 \sin B = 16.7 \sin 125\]

\[\sin B = \frac{16.7 \sin 125}{24.6}\]

\[B = \sin^{-1} \left( \frac{16.7 \sin 125}{24.6} \right)\]

\[B \approx 33.8^\circ\]

Find \(\angle A\).

\[\angle A = 180^\circ - 125^\circ - 33.8^\circ = 21.2^\circ.\]
Find a.

\[
\frac{\sin 125}{24.6} = \frac{\sin 21.2}{a} \quad \text{Solve by cross multiplying.}
\]

\[a \sin 125 = 24.6 \sin 21.2\]
\[a = \frac{24.6 \sin 21.2}{\sin 125}\]
\[a \approx 10.9\]

Therefore, \( \angle C = 86^\circ, a = 8.6, \) and \( b = 16.7. \)

**Example 6:** Given \( \triangle ABC \) with \( \angle B = 55^\circ, a = 12 \) and \( b = 10.5. \) Find \( c, \angle A, \) and \( \angle C. \)

Draw and label the triangle.

\[\text{Determine if the Law of Sines applies. We can use the Law of Sines, to find} \ \angle A.\]

Find \( \angle A. \)

\[
\frac{\sin 55}{10.5} = \frac{\sin A}{12} \quad \text{Solve by cross multiplying.}
\]

\[10.5 \sin A = 12 \sin 55\]
\[\sin A = \frac{12 \sin 55}{10.5}\]
\[A = \sin^{-1} \left( \frac{12 \sin 55}{10.5} \right)\]
\[A \approx 69.4^\circ\]

Thus far everything seems fine, \( \angle A \) should be larger than \( \angle B \) because side \( a \) is larger than side \( b. \) However, \( \sin(69.4^\circ) \approx 0.936059 \) and \( 69.4^\circ \) is not the only answer between \( 0^\circ \) and \( 180^\circ. \) Looking at the graph of \( \sin(x) \) and using the symmetry of the graph, the other possible answer is \( \angle A = 110.6^\circ. \) So, if \( \angle A = 69.4^\circ \) then \( \angle C = 55.6^\circ, \) but if \( \angle A = 110.6 \) then \( \angle C = 14.4^\circ. \) This means this problem has two possible answers and we cannot solve a problem if there are two possible answers. One more example to consider.

**Example 7:** Given \( \triangle ABC \) with \( \angle B = 48^\circ, a = 11 \) and \( b = 13.1. \) Find \( c, \angle A, \) and \( \angle C. \)

Draw and label the triangle.

\[\text{Determine if the Law of Sines applies. We can use the Law of Sines, to find} \ \angle A.\]
Find ∡A.

\[
\frac{\sin 48}{13.1} = \frac{\sin A}{11}
\]

\[13.1 \sin A = 11 \sin 48\]

\[
\sin A = \frac{11 \sin 48}{13.1}
\]

\[A = \sin^{-1}\left(\frac{11 \sin 48}{13.1}\right)\]

\[A \approx 38.6^\circ\]

Find ∡C.

\[\angle C = 180^\circ - 48^\circ - 38.6^\circ = 93.4^\circ\]

Find c.

\[
\frac{\sin 48}{13.1} = \frac{\sin 93.4}{c}
\]

\[c \sin 48 = 13.1 \sin 93.4\]

\[c = \frac{13.1 \sin 93.4}{\sin 48}\]

\[c \approx 17.6\]

Therefore, ∡A = 38.6°, ∡C = 93.4°, and c = 17.6.

When we are given one angle and two sides, the Law of Sines worked in some cases and not in others. In Example 5, the Law of Sines worked because the given angle was 125° and there cannot be any other angles greater than 90° in a triangle. In Example 7, the Law of Sines worked because the next angle we needed to find was smaller than the known angle. In Example 6, the Law of Sines failed because the next angle we needed to find was larger than the known angle.

To summarize, we can use the Law of Sines in the following situations:

- Given two angles and one side.
- Given one angle and sides and the given angle is greater than 90°.
- Given one angle and two sides and the next angle to be found is smaller than the given angle.

In general, the Law of Sines should be used to find all of the smaller angles first!