# Partial Fraction Decomposition

As algebra students we have learned how to add and subtract fractions such as the one show below, but we probably have not been taught how to break the answer back apart into the original question.

$$\frac{4}{x+3}+\frac{2}{x-2}=\frac{4(x-2)+2(x+3)}{(x+3)(x-2)}=\frac{6x-2}{x^2+x-6}$$

The process of breaking a fraction back apart into the original question is called partial fraction decomposition and it is time to learn how to do partial fraction decomposition. Partial fraction decomposition has many different scenarios to pay attention to and we will look at each situation one by one.

**Nonrepeated Linear Factors**

The first scenario we are going to look at is when the denominator has nonrepeated linear factors meaning that when we factor the denominator the factors will be linear and none of the factors will be repeated. Let's show a few examples of what this means:

$x^2 + 3x – 10 = (x + 2)(x – 5)$ is an example of nonrepeated linear factors. Each of the two factors is linear (no higher powers) and each factor occurs only once.

$x^3 – 5x = x(x^2 – 5)$ is not an example of nonrepeated linear factors because $x^2 – 5$ is not linear.

$x^2 – 4x + 4 = (x – 2)(x – 2)$ or $(x – 2)^2$ is not an example of nonrepeated linear factors because the linear factor $(x – 2)$ is repeated twice.

Now that we have an understanding of what a nonrepeated linear factor means. Let's look at how to find the partial fraction decomposition by looking at some examples.

**Example 1** – Find the partial fraction decomposition of $\frac{7x-4}{x^2-2x-8}$.

The first thing we need to do is factor the denominator. Do not worry about factoring the numerator, in this case we cannot factor the numerator, but even if we could factor the numerator we should never do so.

$$\frac{7x-4}{x^2-2x-8}=\frac{7x-4}{(x+2)(x-4)}$$

Now we need to set the problem up correctly. Since this example has nonrepeated linear factors, each of the factors will become denominators in the partial fraction decomposition and the numerator will be unknown at this point. Resulting in:

$$\frac{7x-4}{(x+2)(x-4)}=\frac{A}{x+2}+\frac{B}{x-4}$$

Note: When the factor in the denominator is linear the unknown that goes in the numerator above that factor will be a constant such as A, B, or C. When we look at partial fraction decomposition with nonlinear factors we will see something different in the numerator.

Now that the problem has been set up correctly we need to find the values of A and B. How the values of A and B are found will change depending on how the teacher prefers to do the problem. The technique show here may be different than in other places, but this is how I personally prefer to do partial fraction decomposition.

To find the values of A and B we need to make the fraction go away, this can be accomplished by multiplying the entire problem by the least common denominator (LCD).

$$(x+2)(x-4)\left(\frac{7x-4}{(x+2)(x-4)}=\frac{A}{x+2}+\frac{B}{x-4}\right)$$

$$\rightarrow \quad 7x-4=A(x-4)+B(x+2)$$

$$\rightarrow \quad 7x-4=Ax-4A+Bx+2B$$

From here we need to create a system of equations that we can use to find the values of A and B. Since we need to find two unknowns, A and B, we need to come up with two equations. To find the two equations we group all of the terms that have an x in common and all of the terms that do not have an x in common as follows:

$7x = Ax + Bx \rightarrow A + B = 7.$

$-4 = -4A + 2B \rightarrow -4A + 2B = -4.$

Now we need to solve the system of equations.

$$\begin{array}{r} A+B=7 \\ -4A+2B=-4 \end{array} \rightarrow A=3 \text{ and } B=4$$

Note: The technique used to solve the system of equations will depend on your background in math. You can use substitution, elimination, Cramer's Rule, a graphing calculator, or any other technique you know. Check with your teacher on what technique you can use to solve a system of equations. These notes will assume you can solve any system of linear equations and will not show the details of how the system is solved.

The partial fraction decomposition is $\frac{7x-4}{x^2-2x-8}=\frac{3}{x+2}+\frac{4}{x-4}$.

Summarizing the steps required to find the partial fraction decomposition.

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|---|---|
| Step 1: | Factor the denominator of the fraction. |
| Step 2: | Set the problem up correctly. |
| Step 3: | Multiply by the LCD to make the fractions go away and simplify the result. |
| Step 4: | Create a system of equations. |
| Step 5: | Solve the system of equations. |
| Step 6: | Write the partial fraction decomposition in simplified form. |

These same six steps will be used in all examples. The only difference will be in step 2, setting the problem up correctly. Setting the problem up correctly will depend on what the factors in the denominator look like.

**Example 2** – Find the partial fraction decomposition of $\frac{8}{x^2+11x+28}$.

**Step 1**: Factor the denominator.

$$\frac{8}{x^2+11x+28}=\frac{8}{(x+4)(x+7)}$$

**Step 2**: Set the problem up correctly.

In this case, the denominator has two nonrepeated linear factors so the unknowns will be A and B.

$$\frac{8}{(x+4)(x+7)}=\frac{A}{x+4}+\frac{B}{x+7}$$

**Step 3**: Multiply by the LCD to make the fractions go away and simplify the result.

$$(x+4)(x+7)\left(\frac{8}{(x+4)(x+7)}=\frac{A}{x+4}+\frac{B}{x+7}\right)$$

$$\rightarrow \quad 8=A(x+7)+B(x+4)$$

$$\rightarrow \quad 8=Ax+7A+Bx+4B$$

**Step 4**: Create a system of equations.

Group the terms that have an x in common. Then group the terms that do not have an x in common.

$0x = Ax + Bx \rightarrow A + B = 0$

$8 = 7A + 4B \rightarrow 7A + 4B = 8$

**Step 5**: Solve the system of equations.

$$\begin{array}{r} A+B=0 \\ 7A+4B=8 \end{array} \rightarrow A=\frac{8}{3} \text{ and } B=-\frac{8}{3}$$

**Step 6**: Write the partial fraction decomposition in simplified form.

$$\frac{8}{x^2+11x+28}=\frac{\frac{8}{3}}{x+4}+\frac{-\frac{8}{3}}{x+7}$$

$$=\frac{3}{8(x+4)}-\frac{3}{8(x+7)}$$

Note: To get the simplified version of the answer we multiplied by the reciprocal of x + 4 and x + 7. We also changed the problem to subtraction rather than addition because the value of B was negative.

**Example 3** – Find the partial fraction decomposition of $\frac{3x^2-17x-20}{x^3+3x^2-10x}$.

**Step 1**: Factor the denominator.

$$\frac{3x^2-17x-20}{x^3+3x^2-10x}=\frac{3x^2-17x-20}{x(x-2)(x+5)}$$

**Step 2**: Set the problem up correctly.

In this case, the denominator has three nonrepeated linear factors so the unknowns will be A, B, and C.

$$\frac{3x^2-17x-20}{x(x-2)(x+5)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+5}$$

**Step 3**: Multiply by the LCD to make the fractions go away and simplify the result.

$$x(x-2)(x+5)\left(\frac{3x^2-17x-20}{x(x-2)(x+5)}=\frac{A}{x}+\frac{B}{x-2}+\frac{C}{x+5}\right)$$

$$\rightarrow \quad 3x^2-17x-20=A(x-2)(x+5)+Bx(x+5)+Cx(x-2)$$

$$\rightarrow \quad 3x^2-17x-20=A(x^2+3x-10)+B(x^2+5x)+C(x^2-2x)$$

$$\rightarrow \quad 3x^2-17x-20=Ax^2+3Ax-10A+Bx^2+5Bx+Cx^2-2Cx$$

**Step 4**: Create a system of equations.

Group the terms that have an $x^2$ in common. Group the terms that have x in common. Finally, group the terms that do not have an x in common.

$3x^2 = Ax^2 + Bx^2 + Cx^2 \rightarrow A + B + C = 3$

$-17x = 3Ax + 5Bx - 2Cx \rightarrow 3A + 5B - 2C = -17$

$-20 = -10A \rightarrow -10A = -20$

**Step 5**: Solve the system of equations.

$$\begin{aligned} A+B+C&=3 \\ 3A+5B-2C&=-17 \rightarrow A=2 \text{ and } B=-3 \text{ and } C=4 \\ -10A&=-20 \end{aligned}$$

**Step 6**: Write the partial fraction decomposition in simplified form.

$$\frac{3x^2-17x-20}{x^3+3x^2-10x}=\frac{2}{x}+\frac{-3}{x-2}+\frac{4}{x+5}$$

$$=\frac{2}{x}-\frac{3}{x-2}+\frac{4}{x+5}$$

## Repeated Linear Factors

The next scenario that we are going to consider is repeated linear factors. A repeated linear factor is when a problem factors into the same factor repeated more than once. Some examples are $(x + 1)^2$ or $(x – 5)^3$. When a linear factor is repeated we must adjust the setup of the partial fraction decomposition to account for the repeat(s). When a linear factor is repeated we must create a separate fraction for each time the linear factor is repeated. Let's consider the following problems and look at how we would setup the partial fraction decomposition.

$$\frac{6x+5}{(x-3)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}$$

In this example, we needed two fractions because the linear factor is repeated twice.

$$\frac{x^2-4x+2}{(x+4)^3}=\frac{A}{x+4}+\frac{B}{(x+4)^2}+\frac{C}{(x+4)^3}$$

In this example, we need three fractions because the linear factor is repeated three times.

$$\frac{5x+8}{(x-1)^2(x+3)^2}=\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+3}+\frac{D}{(x+3)^2}$$

In this example, we need four fractions because x – 1 is repeated twice and x + 3 is repeated twice.

Now we are going to work through some examples that have repeated linear factors.

**Example 4** – Find the partial fraction decomposition of $\frac{7x^2+5x+7}{(x-2)(x^2+2x+1)}$.

**Step 1**: Factor the denominator.

$$\frac{7x^2+5x+7}{(x-2)(x^2+2x+1)}=\frac{7x^2+5x+7}{(x-2)(x+1)^2}$$

**Step 2**: Set the problem up correctly.

In this case, the denominator has one nonrepeated linear factor and one repeated linear factor so the unknowns will be A, B, and C.

$$\frac{7x^2+5x+7}{(x-2)(x+1)^2}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}$$

**Step 3**: Multiply by the LCD to make the fractions go away and simplify the result.

$$(x-2)(x+1)^2\left(\frac{7x^2+5x+7}{(x-2)(x+1)^2}=\frac{A}{x-2}+\frac{B}{x+1}+\frac{C}{(x+1)^2}\right)$$

$$\rightarrow\quad 7x^2+5x+7=A(x+1)^2+B(x-2)(x+1)+C(x-2)$$

$$\rightarrow\quad 7x^2+5x+7=A(x^2+2x+1)+B(x^2-x-2)+C(x-2)$$

$$\rightarrow\quad 7x^2+5x+7=Ax^2+2Ax+A+Bx^2-Bx-2B+Cx-2C$$

**Step 4**: Create a system of equations.

Group the terms that have an $x^2$ in common. Group the terms that have x in common. Finally, group the terms that do not have an x in common.

$7x^2 = Ax^2 + Bx^2 \rightarrow A + B = 7$

$5x = 2Ax – Bx + Cx \rightarrow 2A – B + C = 5$

$7 = A – 2B – 2C \rightarrow A – 2B – 2C = 7$

**Step 5**: Solve the system of equations.

$$\begin{aligned} A + B &= 7 \\ 2A - B + C &= 5 \rightarrow A = 5 \text{ and } B = 2 \text{ and } C = -3 \\ A - 2B - 2C &= 7 \end{aligned}$$

**Step 6**: Write the partial fraction decomposition in simplified form.

$$\frac{7x^2 + 5x + 7}{(x-2)(x^2 + 2x + 1)} = \frac{5}{x-2} + \frac{2}{x+1} + \frac{-3}{(x+1)^2}$$

$$= \frac{5}{x-2} + \frac{2}{x+1} - \frac{3}{(x+1)^2}$$

Notice that we used the same steps to find the partial fraction decomposition, the only thing we have to make sure of is that we set the problem up correctly. We need to pay attention to whether or not the linear factors are repeated or not.

**Nonrepeated Irreducible Quadratic Factors**

The next scenario that we are going to look at is nonrepeated irreducible quadratic factors. An irreducible quadratic is a quadratic that cannot be factored, such as $x^2 + 4$ or $x^2 - x + 11$. When we are dealing with an irreducible quadratic it changes the way we set up the partial fraction decomposition. The part that will change is what we put in the numerator. When the numerator is a linear factor we must put a constant such as A, B, or C in the numerator. If the denominator is an irreducible quadratic, we put a linear expression in the numerator such as Ax + B or Cx + D. Consider the following examples.

$$\frac{2x - 5}{(x+1)(x^2 + 4)} = \frac{A}{x+1} + \frac{Bx + C}{x^2 + 4}$$

The first fraction has an A in the numerator because the denominator is a linear factor. The second fraction has Bx + C in the numerator because the denominator is an irreducible quadratic.

$$\frac{x^2 - 4x + 2}{(x^2 + 1)(x^2 - x + 7)} = \frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{x^2 - x + 7}$$

Both fractions have linear expressions in the numerator because each of the denominators contains an irreducible quadratic.

**Example 5** – Find the partial fraction decomposition of $\frac{5x^2-9x+19}{x^3-2x^2+3x-6}$.

**Step 1**: Factor the denominator.

$$\frac{5x^2-9x+19}{x^3-2x^2+3x-6}=\frac{5x^2-9x+19}{(x-2)(x^2+3)}$$

**Step 2**: Set the problem up correctly.

In this case, the denominator has one linear factor and one irreducible quadratic so the unknowns will be A and Bx + C.

$$\frac{5x^2-9x+19}{(x-2)(x^2+3)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}$$

**Step 3**: Multiply by the LCD to make the fractions go away and simplify the result.

$$(x-2)(x^2+3)\left(\frac{5x^2-9x+19}{(x-2)(x^2+3)}=\frac{A}{x-2}+\frac{Bx+C}{x^2+3}\right)$$

$$\rightarrow \quad 5x^2-9x+19=A(x^2+3)+(Bx+C)(x-2)$$

$$\rightarrow \quad 5x^2-9x+19=Ax^2+3A+Bx^2-2Bx+Cx-2C$$

**Step 4**: Create a system of equations.

Group the terms that have an $x^2$ in common. Group the terms that have x in common. Finally, group the terms that do not have an x in common.

$5x^2 = Ax^2 + Bx^2 \rightarrow A + B = 5$

$-9x = -2Bx + Cx \rightarrow -2B + C = -9$

$19 = 3A - 2C \rightarrow 3A - 2C = 19$

**Step 5**: Solve the system of equations.

$$\begin{aligned} A+B&=5 \\ -2B+C&=-9 \rightarrow A=3 \text{ and } B=2 \text{ and } C=-5 \\ 3A-2C&=19 \end{aligned}$$

**Step 6**: Write the partial fraction decomposition in simplified form.

$$\frac{5x^2-9x+19}{x^3-2x^2+3x-6}=\frac{3}{x-2}+\frac{2x-5}{x^2+3}$$

**Example 6** – Find the partial fraction decomposition of $\frac{2x^2+13x-17}{(x+3)(x^2-2x+4)}$.

**Step 1**: Factor the denominator.

In this example the denominator is already factored.

**Step 2**: Set the problem up correctly.
In this case, the denominator has one linear factor and one irreducible quadratic so the unknowns will be A and Bx + C.

$$\frac{2x^2+13x-17}{(x+3)(x^2-2x+4)}=\frac{A}{x+3}+\frac{Bx+C}{x^2-2x+4}$$

**Step 3**: Multiply by the LCD to make the fractions go away and simplify the result.

$$(x+3)(x^2-2x+4)\left(\frac{2x^2+13x-17}{(x+3)(x^2-2x+4)}=\frac{A}{x+3}+\frac{Bx+C}{x^2-2x+4}\right)$$

$$\rightarrow \quad 2x^2+13x-17=A(x^2-2x+4)+(Bx+C)(x+3)$$

$$\rightarrow \quad 2x^2+13x-17=Ax^2-2Ax+4A+Bx^2+3Bx+Cx+3C$$

**Step 4**: Create a system of equations.
Group the terms that have an $x^2$ in common. Group the terms that have x in common. Finally, group the terms that do not have an x in common.

$2x^2 = Ax^2 + Bx^2 \rightarrow A + B = 2$

$13x = -2Ax + 3Bx + Cx \rightarrow -2A + 3B + C = 13$

$-17 = 4A + 3C \rightarrow 4A + 3C = -17$

**Step 5**: Solve the system of equations.

$$\begin{aligned} A+B&=2 \\ -2A+3B+C&=13 \rightarrow A=-2 \text{ and } B=4 \text{ and } C=-3 \\ 4A+3C&=-17 \end{aligned}$$

**Step 6**: Write the partial fraction decomposition in simplified form.

$$\frac{2x^2+13x-17}{(x+3)(x^2-2x+4)}=\frac{-2}{x+3}+\frac{4x-3}{x^2-2x+4}$$

## Repeated Irreducible Quadratic Factors

The final scenario that we are going to consider is repeated irreducible quadratic factors. A repeated irreducible quadratic factor is when a problem has irreducible quadratic factors repeated more than once. Some examples are $(x + 2)^2$ or $(x^2 + 4x - 1)^3$. When an irreducible quadratic factor is repeated we must adjust the setup of the partial fraction decomposition to account for the repeat(s). When an irreducible quadratic factor is repeated we must create a separate fraction for each time the quadratic factor is repeated. Let's consider the following problems and look at how we would setup the partial fraction decomposition.

$$\frac{4x^2+2x-7}{(x^2-3)^2}=\frac{Ax+B}{x^2-3}+\frac{Cx+D}{(x^2-3)^2}$$

Here we needed two fractions because the quadratic factor is repeated twice. Notice linear expression are in the numerator because the denominator is quadratic.

$$\frac{3x^2-7x+9}{(x^2-7)^3}=\frac{Ax+B}{x^2-7}+\frac{Cx+D}{(x^2-7)^2}+\frac{Ex+F}{(x^2-7)^3}$$

Here we need three fractions because the quadratic factor is repeated three times.

$$\frac{11x-6}{(x^2-5x+1)^2}=\frac{Ax+B}{x^2-5x+1}+\frac{Cx+D}{(x^2-5x+1)^2}$$

Here we needed two fractions because the quadratic factor is repeated twice.

## Summary

There are a total of four different cases the we need to be on the lookout for: nonrepeated linear factors, repeated linear factors, nonrepeated irreducible quadratic factors, and repeated irreducible quadratic factors. It is very important that we pay close attention to which case we are dealing with because each case determines how we set the problem up.

- If the denominator contains nonrepeated linear factors, we must create a separate fraction for each linear factor. We also have to put a constant, such as A, B, or C, in the numerator for each of different linear factor.
- If the denominator contains a repeated linear factor, we must create a separate fraction for each time the linear factor is repeated. We also have to put a constant, such as A, B, or C, in the numerator for each of the repeated linear factors.
- If the denominator contains nonrepeated irreducible quadratic factors, we must create a separate fraction for each irreducible quadratic factor. We also have to put a linear expression, such as Ax + B or Dx + E, in the numerator for each irreducible quadratic factor.
- If the denominator contains repeated irreducible quadratic factors, we must create a separate fraction for each time the irreducible quadratic factor is repeated. We also have to put a linear expression, such as Ax + B or Cx + D, in the numerator for each of the repeated irreducible quadratic factors.

**Setting the Problem Up Correctly**

Now we are going to practice setting the problem up correctly. We are not actually going to do the partial fraction decomposition; we are just going to make sure we understand how to set the problem up correctly so that we could do the partial fraction decomposition. To practice all of the steps for partial fraction decomposition try working on the practice problems.

**Example 7**: Set up the partial fraction decomposition of $\frac{5x^2+3x-7}{x(x-4)(x+2)}$.

$$\frac{5x^2+3x-7}{x(x-4)(x+2)}=\frac{A}{x}+\frac{B}{x-4}+\frac{C}{x+2}$$

This problem has three nonrepeated linear factors in the denominator, so we must have three fractions. We place a constant in each of the numerators because each factor in the denominator is linear.

**Example 8**: Set up the partial fraction decomposition of $\frac{9x-5}{x^2(x^2-2)}$.

$$\frac{9x-5}{x^2(x^2-2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{Cx+D}{x^2-2}$$

This problem has one repeated linear factor, $x^2$, in the denominator and one nonrepeated irreducible quadratic, so we must have three fractions. We place constants in the numerator of the first two fractions because the denominators are linear. We place a linear expression in the numerators of the third fraction because the factor in the denominator is quadratic. Note: do not be fooled into thinking that $x^2$ is quadratics it factors into $x \cdot x$ which is a repeated linear factor, the linear factor x is repeated twice.

**Example 9**: Set up the partial fraction decomposition of $\frac{7x^3-4x^2+x-2}{(x-3)^3(x^2+8)^2}$.

$$\frac{7x^3-4x^2+x-2}{(x-3)^3(x^2+8)^2}=\frac{A}{x-3}+\frac{B}{(x-3)^2}+\frac{C}{(x-3)^3}+\frac{Dx+E}{x^2+8}+\frac{Fx+G}{(x^2+8)^2}$$

This problem has one repeated linear factor, $(x-3)^2$, in the denominator and one repeated irreducible quadratic, $(x^2+8)^2$, so we must have five fractions. We place constants in the numerator of the first three fractions because the denominators are linear. We place a linear expression in the numerators of the last two fractions because the factors in the denominator are quadratic.