Writing Logarithms as a Sum, Difference, or Product

What are the Properties of Logarithms?

The properties of logarithms are very similar to the properties of exponents because as we have seen before every exponential equation can be written in logarithmic form and vice versa.

Properties of Logarithms

There are five properties that are used for combining or expanding logarithms. These properties are summarized in the table below. When applying the properties of logarithms in the examples shown below and in future examples, the properties will be referred to by number. The combining of logarithms or writing several logarithms as a single logarithm is often required when solving logarithmic equations.

<table>
<thead>
<tr>
<th>Properties of Logarithms</th>
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<tbody>
<tr>
<td>Property 1: $\log_a 1 = 0$</td>
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<tr>
<td>Property 2: $\log_a a = 1$</td>
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<tr>
<td>Property 3: $\log_a x + \log_a y = \log_a (xy)$</td>
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<td>Property 4: $\log_a x - \log_a y = \log_a \left(\frac{x}{y}\right)$</td>
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<tr>
<td>Property 5: $y \log_a x = \log_a x^y$</td>
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Explanation of the Properties

Property 1: → This property says that log of zero will always equal one, regardless of the base used. If you use your calculator to find $\log(1)$ or $\ln(1)$ the answer will be zero.

Property 2: → This property says that log base a of a will always equal one. If you needed $\log_a 4$, $\log_a 9$, or $\log(10)$ the answer for all three will be one.

Property 3: → This property says that if two logarithms with the same base are being added together then the resulting logarithm will have multiplication.

Property 4: → This property says that if two logarithms with the same base are being subtracted together then the resulting logarithm will have division.

Property 5: → This property says that the number in front of the logarithm will be moved and become the power of the variable.

Now let’s look at a few examples of how to use the properties of logarithms to expand logarithms, that is, how to write a logarithm as a sum, difference, or product of logarithms.
Example 1: Write $\log_3 \left( x^2 y^7 \right)$ as a sum, difference, or product of logarithms.

\[
\log_3 x^2 + \log_3 y^7 \quad \text{Use Property 3 to write as addition.}
\]
\[
2 \log_3 x + 7 \log_3 y \quad \text{Use Property 5 to move the exponents out front.}
\]

Thus, $\log_3 \left( x^2 y^7 \right) = 2 \log_3 x + 7 \log_3 y$.

Example 2: Write $\log_8 \left( \sqrt[3]{x} \right)$ as a sum, difference, or product of logarithms.

\[
\log_8 \left( \frac{x^{\frac{1}{2}}}{y^3} \right) \quad \text{Rewrite the radical using rational exponents (fractions).}
\]
\[
\log_8 \left( x^{\frac{1}{2}} y^{-3} \right) \quad \text{Rewrite using negative exponents.}
\]
\[
\log_8 x^{\frac{1}{2}} + \log_8 y^{-3} \quad \text{Use Property 3 to write as addition.}
\]
\[
\frac{1}{2} \log_8 x - 3 \log_8 y \quad \text{Use Property 5 to move the exponents out front.}
\]

Thus, $\log_8 \left( \sqrt[3]{x} \right) = \frac{1}{2} \log_8 x - 3 \log_8 y$.

Example 3: Write $\log_4 \left( \frac{4x}{y^9} \right)$ as a sum, difference, or product of logarithms.

\[
\log_4 \left( 4xy^{-9} \right) \quad \text{Rewrite using negative exponents.}
\]
\[
\log_4 4 + \log_4 x + \log_4 y^{-9} \quad \text{Use Property 3 to write as addition. (Note: } 4x = 4 \text{ times } x)\]
\[
\log_4 4 + \log_4 x - 9 \log_4 y \quad \text{Use Property 5 to move the exponents out front.}
\]
\[
1 + \log_4 x - 9 \log_4 y \quad \text{Use Property 2 to simplify.}
\]

Thus, $\log_4 \left( \frac{4x}{y^9} \right) = 1 + \log_4 x - 9 \log_4 y$. 

Example 4: Write \( \log \left( \frac{\sqrt[4]{x^3 y^7}}{z^8} \right) \) as a sum, difference, or product of logarithms.

\[
\log \left( \frac{x^{3/4} y^7}{z^8} \right) \quad \text{Rewrite the radical using rational exponents (fractions).}
\]
\[
\log \left( x^{3/4} y^7 z^{-8} \right) \quad \text{Rewrite using negative exponents.}
\]
\[
\log x^{3/4} + \log y^7 + \log z^{-8} \quad \text{Use Property 3 to write as addition.}
\]
\[
\frac{3}{4} \log x + 7 \log y - 8 \log z \quad \text{Use Property 5 to move the exponents out front.}
\]

Thus, \( \log \left( \frac{\sqrt[4]{x^3 y^7}}{z^8} \right) = \frac{3}{4} \log x + 7 \log y - 8 \log z. \)

Example 5: Write \( \ln \left( \frac{x^5}{y^2 z^7} \right) \) as a sum, difference, or product of logarithms.

\[
\ln \left( x^5 y^{-2} z^{-7} \right) \quad \text{Rewrite using negative exponents.}
\]
\[
\ln x^5 + \ln y^{-2} + \ln z^{-7} \quad \text{Use Property 3 to write as addition.}
\]
\[
5 \ln x - 2 \ln y - 7 \ln z \quad \text{Use Property 5 to move the exponents out front.}
\]

Thus, \( \ln \left( \frac{x^5}{y^2 z^7} \right) = 5 \ln x - 2 \ln y - 7 \ln z. \)

Example 6: Write \( \log_8 \left( \frac{1}{\sqrt[5]{xy^4}} \right) \) as a sum, difference, or product of logarithms.

\[
\log_8 \left( \frac{1}{x^{1/5} y^4} \right) \quad \text{Rewrite the radical using rational exponents (fractions).}
\]
\[
\log_8 \left( x^{-1/5} y^{-4} \right) \quad \text{Rewrite using negative exponents.}
\]
\[
\log_8 x^{-1/5} + \log_8 y^{-4} \quad \text{Use Property 3 to write as addition.}
\]
\[
-\frac{1}{5} \log_8 x - 4 \log_8 y \quad \text{Use Property 5 to move the exponents out front.}
\]

Thus, \( \log_8 \left( \frac{1}{\sqrt[5]{xy^4}} \right) = -\frac{1}{5} \log_8 x - 4 \log_8 y. \)