Using Cramer's Rule to Solve Three Equations with Three Unknowns

Here we will be learning how to use Cramer's Rule to solve a linear system with three equations and three unknowns. Cramer's Rule is one of many techniques that can be used to solve systems of linear equations. Cramer's Rule involves the use of determinants to find the solution and like any other technique it has its advantages and disadvantages. Cramer's Rule itself is very simple, but the notation used requires a little bit of explanation, so let's take a look at Cramer's Rule.

Cramer's Rule

$$x = \frac{D_x}{D}, y = \frac{D_y}{D}, z = \frac{D_z}{D}$$

To help explain the notation, consider the following system of equations:

$$\begin{aligned} 2x + 3y - z &= 1 \\ 4x + y - 3z &= 11 \\ 3x - 2y + 5z &= 21 \end{aligned}$$

To find the values of x, y, and z there are four different values that we need to calculate, they are D, D_x, D_y, and D_z. In all four cases the "D" stands for the determinant, now let's look at what they represent.

$$D = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & 5 \end{vmatrix}$$

Here we use the x, y, and z values from the problem to create a 3×3 matrix.

$$D_x = \begin{vmatrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{vmatrix}$$

Here we replace the x-values in the first column with the values after the equal sign and leave the values in the y and z columns unchanged.

$$D_y = \begin{vmatrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{vmatrix}$$

Here we replace the y-values in the second column with the values after the equal sign and leave the values in the x and z columns unchanged.

$$D_z = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{vmatrix}$$

Here we replace the z-values in the third column with the values after the equal sign and leave the values in the x and y columns unchanged.

Once we have calculated the values of D, D_x, D_y, and D_z we can apply Cramer's Rule to find x, y, and z.

To use Cramer's Rule to solve a system of three equations with three unknowns, we need to follow these steps:

Step 1:	Find the determinant, D, by using the x, y, and z values from the problem.
Step 2:	Find the determinant, D_x, by replacing the x-values in the first column with the values after the equal sign leaving the y and z columns unchanged.
Step 3:	Find the determinant, D_y, by replacing the y-values in the second column with the values after the equal sign leaving the x and z columns unchanged.
Step 4:	Find the determinant, D_z, by replacing the z-values in the third column with the values after the equal sign leaving the x and y columns unchanged.
Step 5:	Use Cramer's Rule to find the values of x, y, and z.

Now we are ready to look at a couple of examples. To review how to calculate the determinant of a 3×3 matrix, click here.

Example 1: Use Cramer's Rule to solve
$$\begin{aligned} 2x + 3y - z &= 1 \\ 4x + y - 3z &= 11 \\ 3x - 2y + 5z &= 21 \end{aligned}.$$

Step 1: Find the determinant, D, by using the x, y, and z values from the problem.

$$\begin{matrix} & & -3 \quad 12 \quad 60 \\ D = \begin{vmatrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & 5 \end{vmatrix} = & \left|\begin{matrix} 2 & 3 & -1 \\ 4 & 1 & -3 \\ 3 & -2 & 5 \end{matrix}\right| \begin{matrix} 2 & 3 \\ 4 & 1 \\ 3 & -2 \end{matrix} & = -9 - 69 = -78 \\ & & 10 \quad -27 \quad 8 \end{matrix}$$

Step 2: Find the determinant, D_x, by replacing the x-values in the first column with the values after the equal sign leaving the y and z columns unchanged.

$$\begin{matrix} & & -21 \quad 6 \quad 165 \\ D_x = \begin{vmatrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{vmatrix} = & \left|\begin{matrix} 1 & 3 & -1 \\ 11 & 1 & -3 \\ 21 & -2 & 5 \end{matrix}\right| \begin{matrix} 1 & 3 \\ 11 & 1 \\ 21 & -2 \end{matrix} & = -162 - 150 = -312 \\ & & 5 \quad -189 \quad 22 \end{matrix}$$

Step 3: Find the determinant, D_y, by replacing the y-values in the second column with the values after the equal sign leaving the x and z columns unchanged.

$$\begin{matrix} & & -33 \quad -126 \quad 20 \\ D_y = \begin{vmatrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{vmatrix} = & \left|\begin{matrix} 2 & 1 & -1 \\ 4 & 11 & -3 \\ 3 & 21 & 5 \end{matrix}\right| \begin{matrix} 2 & 1 \\ 4 & 11 \\ 3 & 21 \end{matrix} & = 17 - (-139) = 156 \\ & & 110 \quad -9 \quad -84 \end{matrix}$$

Step 4: Find the determinant, D_z, by replacing the z-values in the third column with the values after the equal sign leaving the x and y columns unchanged.

$$\begin{matrix} & & 3 \quad -44 \quad 252 \\ D_z = \begin{vmatrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{vmatrix} = & \left|\begin{matrix} 2 & 3 & 1 \\ 4 & 1 & 11 \\ 3 & -2 & 21 \end{matrix}\right| \begin{matrix} 2 & 3 \\ 4 & 1 \\ 3 & -2 \end{matrix} & = 133 - 211 = -78 \\ & & 42 \quad 99 \quad -8 \end{matrix}$$

Step 5: Use Cramer's Rule to find the values of x, y, and z.

$$x = \frac{D_x}{D} = \frac{-312}{-78} = 4$$

$$y = \frac{D_y}{D} = \frac{156}{-78} = -2$$

$$z = \frac{D_z}{D} = \frac{-78}{-78} = 1$$

Generally, the answer is written as on order triple (4, –2, 1) representing the x, y, and z values.

Example 2: Use Cramer's Rule to solve $\begin{array}{c} -x + 3y - 2z = 5 \\ 4x - y - 3z = -8 \\ 2x + 2y - 5z = 7 \end{array}$.

Step 1: Find the determinant, D, by using the x, y, and z values from the problem.

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad 4 \quad 6 \quad -60 \end{array}$$

$$D = \begin{vmatrix} -1 & 3 & -2 \\ 4 & -1 & -3 \\ 2 & 2 & -5 \end{vmatrix} = \left|\begin{array}{ccc|cc} -1 & 3 & -2 & -1 & 3 \\ 4 & -1 & -3 & 4 & -1 \\ 2 & 2 & -5 & 2 & 2 \end{array}\right. = -39 - (-50) = 11$$

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad -5 \quad -18 \quad -16 \end{array}$$

Step 2: Find the determinant, D_x, by replacing the x-values in the first column with the values after the equal sign leaving the y and z columns unchanged.

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad 14 \quad -30 \quad 120 \end{array}$$

$$D_x = \begin{vmatrix} 5 & 3 & -2 \\ -8 & -1 & -3 \\ 7 & 2 & -5 \end{vmatrix} = \left|\begin{array}{ccc|cc} 5 & 3 & -2 & 5 & 3 \\ -8 & -1 & -3 & -8 & -1 \\ 7 & 2 & -5 & 7 & 2 \end{array}\right. = -6 - 104 = -110$$

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad 25 \quad -63 \quad 32 \end{array}$$

Step 3: Find the determinant, D_y, by replacing the y-values in the second column with the values after the equal sign leaving the x and z columns unchanged.

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad 32 \quad 21 \quad -100 \end{array}$$

$$D_y = \begin{vmatrix} -1 & 5 & -2 \\ 4 & -8 & -3 \\ 2 & 7 & -5 \end{vmatrix} = \left|\begin{array}{ccc|cc} -1 & 5 & -2 & -1 & 5 \\ 4 & -8 & -3 & 4 & -8 \\ 2 & 7 & -5 & 2 & 7 \end{array}\right. = -126 - (-47) = -79$$

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad -40 \quad -30 \quad -56 \end{array}$$

Step 4: Find the determinant, D_z, by replacing the z-values in the third column with the values after the equal sign leaving the x and y columns unchanged.

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad -10 \quad 16 \quad 84 \end{array}$$

$$D_z = \begin{vmatrix} -1 & 3 & 5 \\ 4 & -1 & -8 \\ 2 & 2 & 7 \end{vmatrix} = \left|\begin{array}{ccc|cc} -1 & 3 & 5 & -1 & 3 \\ 4 & -1 & -8 & 4 & -1 \\ 2 & 2 & 7 & 2 & 2 \end{array}\right. = -1 - 90 = -91$$

$$\begin{array}{c} \quad\quad\quad\quad\quad\quad\quad\quad\quad 7 \quad -48 \quad 40 \end{array}$$

Step 5: Use Cramer's Rule to find the values of x, y, and z.

$$x = \frac{D_x}{D} = \frac{-110}{11} = -10$$

$$y = \frac{D_y}{D} = \frac{-79}{11} = -\frac{79}{11}$$

$$z = \frac{D_z}{D} = \frac{-91}{11} = -\frac{91}{11}$$

Answer written as on order triple: $\left(-10, -\frac{79}{11}, -\frac{91}{11}\right)$

Advantages and Disadvantages of Cramer's Rule

Advantages – I find that one of the advantages to Cramer's Rule is that you can find the value of x, y, or z without having to know any of the other values of x, y, or z. For example, if you needed to find just the value of y, Cramer's Rule would work well. Another thing that I like about Cramer's Rule is that if any of the values of x, y, or z is are fractions, you do not have to plug in a fraction to find the other values. Each value can be found independently.

Disadvantages – One of the only disadvantages to using Cramer's rule is if the value of D is zero then Cramer's Rule will not work because you cannot divide by zero. However, if the value of D is zero then you know that the solution is either "No Solution" or "Infinite Solutions". You will have to use a different technique such as Addition/Elimination to find out whether the answer is "No Solution" or "Infinite Solutions".