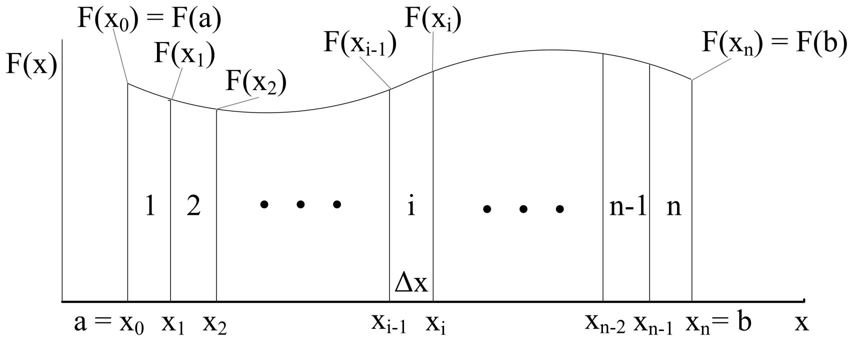
Proof of the Fundamental Theorem of Calculus

Given: $F(x) = \int_a^b f(x)\,dx$



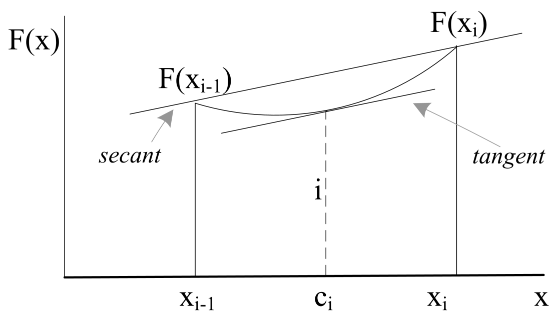
Then

$$F(b) - F(a) = \overbrace{F(x_n)}^{F(b)} - F(x_{n-1}) + F(x_{n-1}) - F(x_{n-2}) + F(x_{n-2}) - \cdots$$
$$- F(x_3) + F(x_3) - F(x_2) + F(x_2) - F(x_1) + F(x_1) - \underbrace{F(x_0)}_{F(a)} \quad (1)$$

$$F(b) - F(a) = \sum_{i=1}^{n} F(x_i) - \sum_{i=1}^{n} F(x_{i-1}) \quad (2)$$

$$= \sum_{i=1}^{n} [F(x_i) - F(x_{i-1})] \quad (3)$$

Now look at the i^{th} interval:



By the Mean Value Theorem, the slope of the tangent line = the slope of the secant line. Therefore,

$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}} \quad (4)$$

$$= \frac{F(x_i) - F(x_{i-1})}{\Delta x} \quad (5)$$

If $\int_a^b f(x)\,dx = F(x)$ then $F'(c_i) = f(c_i)$, so $f(c_i)\,\Delta x = F(x_i) - F(x_{i-1})$. In summing all the partitions,

$$\begin{aligned} \sum_{i=1}^{n} f(c_i)\,\Delta x &= \sum_{i=1}^{n} [F(x_i) - F(x_{x-1})] && (6) \\ &= F(b) - F(a) && (7) \end{aligned}$$

Then take the $\lim_{n\to\infty}$ on both sides:

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{n\to\infty} [F(b) - F(a)] \tag{8}$$

As $n \to \infty$, then $\Delta x \to 0$ and $c_i \to x_i$ so $f(c_i) \to f(x_i)$. Therefore,

$$\lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(c_i)\,\Delta x = \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x = \lim_{n\to\infty} [F(b) - F(a)] \tag{9}$$

Because $F(b)$ and $F(a)$ are constants, $\lim_{n\to\infty}[F(b) - F(a)] = F(b) - F(a)$, so

$$\begin{aligned} \lim_{\substack{n\to\infty \\ \Delta x\to 0}} \sum_{i=1}^{n} f(x_i)\,\Delta x &= F(b) - F(a) && (10) \\ &= \int_a^b f(x)\,dx && (11) \end{aligned}$$

In conclusion,

$$\lim_{n\to\infty \text{ or } \Delta x\to 0} \sum_{i=1}^{n} f(x_i)\,\Delta x = \int_a^b f(x)\,dx \tag{12}$$