

MAT 241- Calculus 3- Prof. Santilli
Toughloves Chapter 15

1.) Midpoint approximation of an iterated integral: $\iint_R f(x, y) dA = \sum_{i=1}^m \sum_{j=1}^n f(\bar{x}_i, \bar{y}_j) \Delta A$, where \bar{x}_i is the midpoint of the x_i th interval and \bar{y}_j is the midpoint of the y_j th interval.

2.) Average value of a function: $f_{avg} = \frac{1}{A_R} \iint_R f(x, y) dA$.

3.) Area of a region: $A = \iint_R dA = \int_{y=c}^{y=d} \int_{x=a}^{x=b} dx dy$ in rectangular coordinates.

4.) Area of a region: $A = \iint_R dA = \int_{\alpha}^{\beta} \int_a^b r dr d\theta$ in polar coordinates.

5.) Volume of a solid: $V = \iint_R f(x, y) dA = \iint_R f(x, y) dx dy$ in rectangular coordinates.

6.) Volume of a solid: $V = \iint_R f(r, \theta) dA = \int_{\alpha}^{\beta} \int_a^b f(r, \theta) r dr d\theta$ in polar coordinates.

7.) Planar lamina of variable density: Examples

Total Mass, $m = \iint_R \rho(x, y) dA$ where $\rho(x, y)$ is the mass density

Total Charge, $Q = \iint_R \sigma(x, y) dA$ where $\sigma(x, y)$ is the charge density

Total Population, $P = \iint_R \delta(x, y) dA$ where $\delta(x, y)$ is the population density

8.) Torque or Force Moments (First Moments) for planar lamina of variable density:

Torque about the y-axis, $\tau_y = \iint_R xg(x, y)\rho(x, y) dA$

Torque about the x-axis, $\tau_x = \iint_R yg(x, y)\rho(x, y) dA$

9.) Mass Moments (First Moments) for planar lamina of variable density:

Moment about the y-axis, $M_y = \iint_R x\rho(x, y) dA$

Moment about the x-axis, $M_x = \iint_R y\rho(x, y) dA$

10.) Geometric Moments (First Moments) for planar lamina:

Moment about the y-axis, $G_y = \iint x dA$

Moment about the x-axis, $G_x = \iint_R y dA$

11.) Center of Gravity for planar lamina of variable density:

$$\bar{x}_g = \frac{\tau_y}{F} = \frac{\iint_R xg(x,y)\rho(x,y)dA}{\iint_R g(x,y)\rho(x,y)dA} \quad \bar{y}_g = \frac{\tau_x}{F} = \frac{\iint_R yg(x,y)\rho(x,y)dA}{\iint_R g(x,y)\rho(x,y)dA}$$

12.) Center of Mass for planar lamina of variable density:

$$\bar{x}_m = \frac{M_y}{m} = \frac{\iint_R x\rho(x,y)dA}{\iint_R \rho(x,y)dA} \quad \bar{y}_m = \frac{M_x}{m} = \frac{\iint_R y\rho(x,y)dA}{\iint_R \rho(x,y)dA}$$

13.) Centroid for planar lamina:

$$\bar{x}_c = \frac{G_y}{A} = \frac{\iint_R x dA}{\iint_R dA} \quad \bar{y}_c = \frac{G_x}{A} = \frac{\iint_R y dA}{\iint_R dA}$$

14.) Moments of Inertia (Second Moments) for planar lamina of variable density:

Moment about the x-axis, $I_x = \iint_R y^2 \rho(x,y) dA$

Moment about the y-axis, $I_y = \iint_R x^2 \rho(x,y) dA$

Moment about the origin (polar moment),

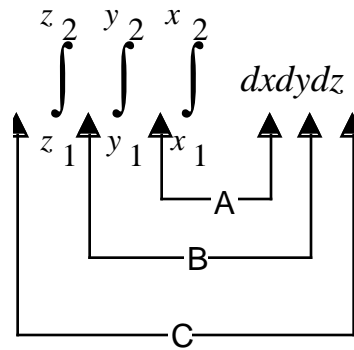
$$I_o = \iint_R r^2 \rho(x,y) dA = \iint_R (x^2 + y^2) \rho(x,y) dA = I_x + I_y$$

15.) Radius of Gyration for planar lamina of variable density:

$$\bar{x} = \sqrt{\frac{I_y}{m}}, \quad \bar{y} = \sqrt{\frac{I_x}{m}}, \quad \bar{r} = \sqrt{\frac{I_o}{m}} = \sqrt{\frac{I_y + I_x}{m}} = \sqrt{\bar{x}^2 + \bar{y}^2}$$

16.) Surface Area: $S = \iint_{surface} |\bar{a} \times \bar{b}| = \iint_R \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1} dA$

17.) Triple Integrals:



A= sweep in the x direction \rightarrow line (length)

B= sweep in the y direction \rightarrow plane (area)

C= sweep in the z direction \rightarrow space (volume)

18.) For 3-D Solids: Total Mass, $m = \iiint_{solid} \rho(x, y, z) dV$ where $\rho(x, y, z)$ is the mass density

19.) Mass Moments (First Moments) for 3-D Solids of variable density:

$$\text{Moment about the y-z plane, } M_{yz} = \iiint_{solid} x\rho(x, y, z) dV$$

$$\text{Moment about the x-z plane, } M_{xz} = \iiint_{solid} y\rho(x, y, z) dV$$

$$\text{Moment about the x-y plane, } M_{xy} = \iiint_{solid} z\rho(x, y, z) dV$$

20.) Center of Mass for 3-D Solids of variable density:

$$\bar{x}_m = \frac{M_{yz}}{m}, \quad \bar{y}_m = \frac{M_{xz}}{m}, \quad \bar{z}_m = \frac{M_{xy}}{m}$$

21.) Moments of Inertia (Second Moments) for 3-D Solids of variable density:

$$\text{Moment about the x-axis, } I_x = \iiint_{Solid} (y^2 + z^2) \rho(x, y, z) dV$$

$$\text{Moment about the y-axis, } I_y = \iiint_{Solid} (x^2 + z^2) \rho(x, y, z) dV$$

$$\text{Moment about the z-axis, } I_z = \iiint_{Solid} (x^2 + y^2) \rho(x, y, z) dV$$

OR-

$$\text{Moment about the x-axis, } I_x = I_{xy} + I_{xz}$$

$$\text{Moment about the y-axis, } I_y = I_{yx} + I_{yz}$$

$$\text{Moment about the z-axis, } I_z = I_{zx} + I_{zy}$$

$$\text{Where } I_{xy} = \iiint_{Solid} z^2 \rho dV, \quad I_{xz} = \iiint_{Solid} y^2 \rho dV, \quad I_{yz} = \iiint_{Solid} x^2 \rho dV$$

22.) Cylindrical Coordinates:

$$x = r \cos \theta, y = r \sin \theta, z = z \text{ and } x^2 + y^2 = r^2, \tan \theta = \frac{y}{x}, z = z.$$

$$23.) V = \iiint_S dx dy dz = \iiint_S r dr d\theta dz \text{ where } dV = dx dy dz = r dr d\theta dz$$

24.) Spherical Coordinates:

$$\begin{aligned} & \rho \geq 0 \\ x = \rho \sin \phi \cos \theta, y = \rho \sin \phi \sin \theta, z = \rho \cos \phi \text{ and } 0 \leq \theta \leq 2\pi \\ & 0 \leq \phi \leq \pi \end{aligned}$$

$$25.) V = \iiint_S dx dy dz = \iiint_S \rho^2 \sin \phi d\rho d\phi d\theta \text{ where } dV = \rho^2 \sin \phi d\rho d\phi d\theta$$

$$26.) \text{ Jacobian for 2-D: } J = \frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u} \frac{\partial y}{\partial v} - \frac{\partial x}{\partial v} \frac{\partial y}{\partial u}$$

27.) To map $f(x, y)$ on R_1 onto $f(g(u, v), h(u, v))$ on R_2 where the transformations are $x = g(u, v)$ and $y = h(u, v)$:

$$dA = dx dy = |J| du dv \text{ therefore } \iint_{R_1} f(x, y) dx dy = \iint_{R_2} f(g(u, v), h(u, v)) |J| du dv$$

$$28.) \text{ Jacobian for 3-D: } J = \frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

29.) To map $f(x, y, z)$ on R_1 onto $f(g(u, v, w), h(u, v, w), j(u, v, w))$ on R_2 where the transformations are $x = g(u, v, w)$, $y = h(u, v, w)$ and $z = j(u, v, w)$:

$$dV = dx dy dz = |J| du dv dw \text{ therefore}$$

$$\iiint_{R_1} f(x, y, z) dV = \iiint_{R_2} f(g(u, v, w), h(u, v, w), j(u, v, w)) |J| du dv dw$$