# MAT 241- Calculus 3- Prof. Santilli
## Toughloves Chapter 14

1.) Level curves (contour curves) are the cross sections of the surface $z = z(x, y)$ projected onto the x-y plane, i.e., $z = z(x, y) = \text{constant}$.

2.) Level surfaces (contour surfaces) are the cross surfaces of the 4-D function, $h = h(x, y, z)$ projected onto the x-y–z space, i.e., $h = h(x, y, z) = \text{constant}$.

3.) A function is continuous at $(x_o, y_o)$ if $f(x_o, y_o) = \lim\limits_{(x,y)\to(x_o,y_o)} f(x, y)$ for all paths.

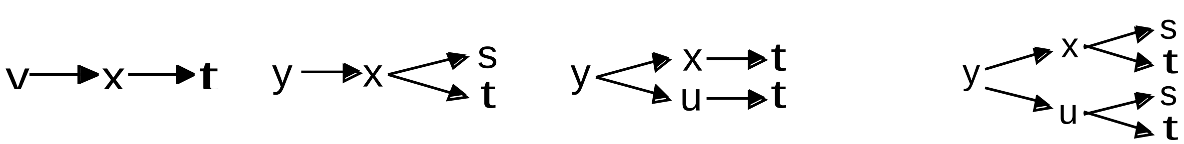
4.) Definition of Partial Derivatives: $f_x = f_x(x,y) = \dfrac{\partial f}{\partial x} = \dfrac{\partial f(x,y)}{\partial x} = \dfrac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$

$$f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x} \quad \text{and} \quad f_y(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}$$

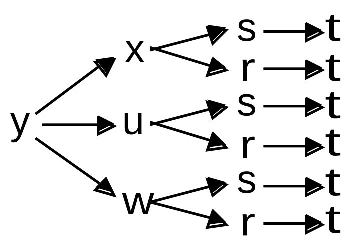
5.) $\dfrac{\partial f(x,y)}{\partial x}$ is the slope of the tangent line to the surface $f(x, y)$ in the x-direction.

6.) $\dfrac{\partial f(x,y)}{\partial y}$ is the slope of the tangent line to the surface $f(x, y)$ in the y-direction.

7.) Chain Rule:




$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \qquad \frac{\partial y}{\partial t} = \frac{dy}{dx}\frac{\partial x}{\partial t} \qquad \frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{dx}{dt} + \frac{\partial y}{\partial u}\frac{du}{dt} \qquad \frac{\partial y}{\partial t} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial u}\frac{\partial u}{\partial t}$$




$$\frac{dy}{dt} = \frac{\partial y}{\partial x}\frac{\partial x}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial x}\frac{\partial x}{\partial s}\frac{ds}{dt} + \frac{\partial y}{\partial u}\frac{\partial u}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial u}\frac{\partial u}{\partial s}\frac{ds}{dt} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial r}\frac{dr}{dt} + \frac{\partial y}{\partial w}\frac{\partial w}{\partial s}\frac{ds}{dt}$$

8.) Higher Order Partials:

$$f_{xx} = f_{xx}(x,y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = f_{11} = D_{11} f = D_{xx} f$$

$$f_{yy} = f_{yy}(x,y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} = f_{22} = D_{22} f = D_{22} f$$

$$f_{xy} = f_{xy}(x,y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f(x,y)}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = f_{12} = D_{12} f = D_{xy} f$$

9.) If the function has continuous second partials, $f_{xy} = f_{yx}$

10.) Implicit Function Theorem:

$$\frac{dy}{dx} = \frac{-f_x}{f_y} \text{ for function of 2 variables}$$

$$\frac{\partial z}{\partial x} = \frac{-f_x}{f_z} \text{ for function of 3 variables}$$

11.) Equation of a tangent plane is:

$$z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)$$

12.) Total Differential: $dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

13.) Gradient Vector: $\overline{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \langle f_x, f_y \rangle$ in plane

$$\overline{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = \langle f_x, f_y, f_z \rangle \text{ in space}$$

14.) Directional Derivative: $D_{\hat{u}} f = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta = \overline{\nabla} f \bullet \hat{u}$ which is the slope of the tangent line to the function in the direction of the unit vector $\hat{u} = \langle \cos\theta, \sin\theta \rangle$.

15.) $D_{\hat{u}} f_{\max} = |\overline{\nabla} f|$ is the max value of the directional derivative and it occurs in the direction of the gradient vector- STEEPEST ASCENT.

16.) $D_{\hat{u}} f_{\min} = -|\overline{\nabla} f|$ is the min. value of the directional derivative and it occurs in the opposite direction of the gradient vector- STEEPEST DECENT.

17.) $\overline{\nabla} f$ of a function is always normal to the level curves or level surfaces of the function at any point on the level curve or level .

18.) Given the function $z = f(x, y)$, then $F(x, y, z) = f(x, y) - z = 0$ is a level surface of $F(x, y, z)$.

19.) Equation of tangent plane to a surface $z = f(x, y)$ at $P(x_o, y_o, z_o)$:

$$\frac{\partial F}{\partial x}(x_o, y_o, z_o)(x - x_o) + \frac{\partial F}{\partial y}(x_o, y_o, z_o)(y - y_o) + \frac{\partial F}{\partial z}(x_o, y_o, z_o)(z - z_o) = 0 \text{ where}$$

$F(x, y, z) = f(x, y) - z$

20.) Equation of the normal line to the surface $z = f(x,y)$ at $P(x_o, y_o, z_o)$:

$$\frac{x - x_o}{F_x(x_o, y_o, z_o)} = \frac{y - y_o}{F_y(x_o, y_o, z_o)} = \frac{z - z_o}{F_z(x_o, y_o, z_o)} \text{ where } F(x,y,z) = f(x,y) - z$$

21.) Normal vector the a surface: $\bar{n} = n_x\hat{i} + n_y\hat{j} + n_z\hat{k} = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$ where $F(x,y,z) = f(x,y) - z$

22.) Angle of inclination of a tangent plane to a surface:

$$\cos\theta = \frac{|n_z|}{|\bar{n}|} = \frac{|F_z|}{|\bar{\nabla} F|} \text{ where } F(x,y,z) = f(x,y) - z$$

23.) Relative extrema in $f(x,y)$ occur at the critical points.

24.) Absolute extrema in $f(x,y)$ occur at the critical points or at the boundary points of the domain.

25.) Local extrema- critical points at point (a,b): $\bar{\nabla} f = \bar{0}$ or DNE.

26.) 3 possible types of critical points: Local max, local min, and saddle.

27.) Second Partial Test- To determine the type of CP at (a,b): Find $d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix}$,

If d > 0 and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum
If d > 0 and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum
If d < 0 then (a, b, f(a,b)) is a saddle point
If d = 0 then test inconclusive.

28.) Constrained optimization with Lagrange Multiplier: optimization occurs where the objective level curves = constraint level curves so $\bar{\nabla} A(x,y) = \lambda \bar{\nabla} g(x,y)$, where $A(x,y)$ is the objective level curve, $g(x,y)$ is the constraint level curve and $\lambda$ is the Larange Multiplier.

29.) Constrained optimization with 2 constraints: $\bar{\nabla} A(x,y) = \lambda \bar{\nabla} g(x,y) + \mu \bar{\nabla} h(x,y)$, where $A(x,y)$ is the objective level curve, $g(x,y)$ and $h(x,y)$ is the constraint level curve and $\lambda$ and $\mu$ are the Lagrange Multipliers.