1.) Level curves (contour curves) are the cross sections of the surface \( z = z(x, y) \) projected onto the x-y plane, i.e., \( z = z(x, y) = \text{constant} \).

2.) Level surfaces (contour surfaces) are the cross surfaces of the 4-D function, \( h = h(x, y, z) \) projected onto the x-y–z space, i.e., \( h = h(x, y, z) = \text{constant} \).

3.) A function is continuous at \((x_0, y_0)\) if \( f(x_0, y_0) = \lim_{(x,y) \to (x_0, y_0)} f(x, y) \) for all paths.

4.) Definition of Partial Derivatives: \( f_x = f_x(x, y) = \frac{\partial f}{\partial x} = \frac{\partial f(x, y)}{\partial x} = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f \)

\[
 f_x(x, y) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}
\]

and \( f_y(x, y) = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \)

5.) \( \frac{\partial f(x, y)}{\partial x} \) is the slope of the tangent line to the surface \( f(x, y) \) in the x-direction.

6.) \( \frac{\partial f(x, y)}{\partial y} \) is the slope of the tangent line to the surface \( f(x, y) \) in the y-direction.

7.) Chain Rule:

\[
 v \xrightarrow{\mathbf{v}} x \xrightarrow{\mathbf{s}} t \quad y \xrightarrow{\mathbf{y}} x \xrightarrow{\mathbf{t}} s \quad \mathbf{y} \xrightarrow{\mathbf{u}} \mathbf{t} \quad \mathbf{y} \xrightarrow{\mathbf{u}} \mathbf{t}
\]

\[
 \frac{dy}{dt} = \frac{dy}{dx} \frac{dx}{dt} \quad \frac{\partial y}{\partial t} = \frac{dy}{dx} \frac{\partial x}{\partial t} \quad \frac{dy}{dt} = \frac{\partial y}{\partial x} \frac{dx}{dt} + \frac{\partial y}{\partial u} \frac{du}{dt} \quad \frac{\partial y}{\partial t} = \frac{\partial y}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial y}{\partial u} \frac{\partial u}{\partial t}
\]

8.) Higher Order Partials:

\[
 f_{xx} = f_{xx}(x, y) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f(x, y)}{\partial x^2} = \frac{\partial^2 z}{\partial x^2} = f_{11} = D_{11} f = D_{xx} f
\]

\[
 f_{yy} = f_{yy}(x, y) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 f(x, y)}{\partial y^2} = \frac{\partial^2 z}{\partial y^2} = f_{22} = D_{22} f = D_{yy} f
\]

\[
 f_{xy} = f_{xy}(x, y) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f(x, y)}{\partial y \partial x} = \frac{\partial^2 z}{\partial y \partial x} = f_{12} = D_{12} f = D_{xy} f
\]
9.) If the function has continuous second partials, $f_{xy} = f_{yx}$

10.) Implicit Function Theorem:
\[
\begin{align*}
\frac{dy}{dx} &= -\frac{f_x}{f_y} & \text{for function of 2 variables} \\
\frac{\partial z}{\partial x} &= -\frac{f_x}{f_z} & \text{for function of 3 variables}
\end{align*}
\]

11.) Equation of a tangent plane is:
\[
z - z_o = f_x(x_o, y_o)(x - x_o) + f_y(x_o, y_o)(y - y_o)
\]

12.) Total Differential: $dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$

13.) Gradient Vector: $\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} = \{f_x, f_y\}$ in plane
\[
\nabla f = \frac{\partial f}{\partial x} \hat{i} + \frac{\partial f}{\partial y} \hat{j} + \frac{\partial f}{\partial z} \hat{k} = \{f_x, f_y, f_z\} \quad \text{in space}
\]

14.) Directional Derivative: $D_{\hat{u}} f = \frac{\partial f}{\partial x} \cos \theta + \frac{\partial f}{\partial y} \sin \theta = \nabla f \cdot \hat{u}$ which is the slope of the tangent line to the function in the direction of the unit vector $\hat{u} = \langle \cos \theta, \sin \theta \rangle$.

15.) $D_{\hat{u}} f_{\text{max}} = |\nabla f|$ is the max value of the directional derivative and it occurs in the direction of the gradient vector- STEEPEST ASCENT.

16.) $D_{\hat{u}} f_{\text{min}} = -|\nabla f|$ is the min. value of the directional derivative and it occurs in the opposite direction of the gradient vector- STEEPEST DECENT.

17.) $\nabla f$ of a function is always normal to the level curves or level surfaces of the function at any point on the level curve or level.

18.) Given the function $z = f(x, y)$, then $F(x, y, z) = f(x, y) - z = 0$ is a level surface of $F(x, y, z)$.

19.) Equation of tangent plane to a surface $z = f(x, y)$ at $P(x_o, y_o, z_o)$:
\[
\frac{\partial F}{\partial x}(x_o, y_o, z_o)(x - x_o) + \frac{\partial F}{\partial y}(x_o, y_o, z_o)(y - y_o) + \frac{\partial F}{\partial z}(x_o, y_o, z_o)(z - z_o) = 0 \quad \text{where} \\
F(x, y, z) = f(x, y) - z
\]
20.) Equation of the normal line to the surface \( z = f(x,y) \) at \( P(x_0,y_0,z_0) \):
\[
\frac{x-x_0}{F_x(x_0,y_0,z_0)} = \frac{y-y_0}{F_y(x_0,y_0,z_0)} = \frac{z-z_0}{F_z(x_0,y_0,z_0)}
\]
where \( F(x,y,z) = f(x,y) - z \)

21.) Normal vector the a surface:
\[
\overrightarrow{n} = n_x \hat{i} + n_y \hat{j} + n_z \hat{k} = \frac{\partial F}{\partial x} \hat{i} + \frac{\partial F}{\partial y} \hat{j} + \frac{\partial F}{\partial z} \hat{k}
\]
where \( F(x,y,z) = f(x,y) - z \)

22.) Angle of inclination of a tangent plane to a surface:
\[
\cos \theta = \frac{|n_z|}{|\nabla F|} = \frac{|F_z|}{|\nabla F|}
\]
where \( F(x,y,z) = f(x,y) - z \)

23.) Relative extrema in \( f(x,y) \) occur at the critical points.

24.) Absolute extrema in \( f(x,y) \) occur at the critical points or at the boundary points of the domain.

25.) Local extrema- critical points at point \((a,b)\): \( \nabla f = \overrightarrow{0} \) or DNE.

26.) 3 possible types of critical points: Local max, local min, and saddle.

27.) Second Partial Test- To determine the type of CP at \((a,b)\): Find \( d = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{yx}(a,b) & f_{yy}(a,b) \end{vmatrix} \)

- If \( d > 0 \) and \( f_{xx}(a,b) > 0 \) then \((a,b)\) is a local minimum
- If \( d > 0 \) and \( f_{xx}(a,b) < 0 \) then \((a,b)\) is a local maximum
- If \( d < 0 \) then \((a, b, f(a,b))\) is a saddle point
- If \( d = 0 \) then test inconclusive.

28.) Constrained optimization with Lagrange Multiplier: optimization occurs where the objective level curves = constraint level curves so \( \nabla A(x,y) = \lambda \nabla g(x,y) \), where \( A(x,y) \) is the objective level curve, \( g(x,y) \) is the constraint level curve and \( \lambda \) is the Lagrange Multiplier.

29.) Constrained optimization with 2 constraints: \( \nabla A(x,y) = \lambda \nabla g(x,y) + \mu \nabla h(x,y) \), where \( A(x,y) \) is the objective level curve, \( g(x,y) \) and \( h(x,y) \) is the constraint level curve and \( \lambda \) and \( \mu \) are the Lagrange Multipliers.