MAT 241- Calculus 3- Prof. Santilli
Toughloves Chapter 13

1.) Vector-value function: $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j}$: plane curve in $\Re^2$

$\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$: Space curve in $\Re^3$

Domain: Intersection of the vector function component's domains.

2.) Tangent vector to the curve is $\bar{T}(t) = \bar{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$

3.) Unit tangent vector to the curve is $\hat{T}(t) = \dfrac{\bar{T}(t)}{|\bar{T}(t)|} = \dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$

4.) $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$ is smooth if $x'(t), y'(t), z'(t)$ are all continuous and $\bar{r}'(t) \neq 0$.

5.) Piecewise smooth = curve that is made up of a finite number of smooth pieces.

6.) $\dfrac{d(\bar{u}(t) \pm \bar{v}(t))}{dt} = \dfrac{d\bar{u}(t)}{dt} \pm \dfrac{d\bar{v}(t)}{dt} = \bar{u}'(t) \pm \bar{v}'(t)$

7.) $\dfrac{d(c\bar{u}(t))}{dt} = c\dfrac{d\bar{u}(t)}{dt} = c\bar{u}'(t)$

8.) Product rule (scalar funct)(vector)

$$\frac{d(f(t)\bar{u}(t))}{dt} = \frac{df(t)}{dt}\bar{u}(t) + f(t)\frac{d\bar{u}(t)}{dt} = f'(t)\bar{u}(t) + f(t)\bar{u}'(t)$$

9.) Product rule (vect)•(vect)

$$\frac{d(\bar{u}(t) \bullet \bar{v}(t))}{dt} = \frac{d\bar{u}(t)}{dt} \bullet \bar{v}(t) + \bar{u}(t) \bullet \frac{d\bar{v}(t)}{dt} = \bar{u}'(t) \bullet \bar{v}(t) + \bar{u}(t) \bullet \bar{v}'(t)$$

10.) Product rule (vect)x(vect) – ORDER IS IMPORTANT!

$$\frac{d(\bar{u}(t) \times \bar{v}(t))}{dt} = \frac{d\bar{u}(t)}{dt} \times \bar{v}(t) + \bar{u}(t) \times \frac{d\bar{v}(t)}{dt} = \bar{u}'(t) \times \bar{v}(t) + \bar{u}(t) \times \bar{v}'(t)$$

11.) Chain rule

$$\frac{d[\bar{u}(f(t))]}{dt} = \frac{d[\bar{u}(f(t))]}{df}\frac{d[f(t)]}{dt} = \bar{u}'(f)f'(t)$$

12.) ***<u>IMPORTANT!</u>***

If $\bar{r}(t) \bullet \bar{r}(t) = \text{constant}$, then $\bar{r}(t) \bullet \bar{r}'(t) = 0$,

which means $|\bar{r}(t)| = \text{constant}$, then $\bar{r}(t) \perp \bar{r}'(t)$ or $\bar{r}(t) \perp \bar{T}(t)$.

NOTE:

For any **UNIT VECTOR**, $|\hat{r}(t)| = \sqrt{\hat{r}(t) \bullet \hat{r}(t)} = 1 = \text{constan}$, and thus $\hat{r}(t) \bullet \hat{r}'(t) = 0$

which means $\hat{r}(t) \perp \hat{r}'(t)$. ***Any unit vector is perpendicular to its derivative vector.***

13.) For $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

$$\int \bar{r}(t)dt = \int x(t)dt\,\hat{i} + \int y(t)dt\,\hat{j} + \int z(t)dt\,\hat{k}$$ and

$$\int_a^b \bar{r}(t)dt = \int_a^b x(t)dt\,\hat{i} + \int_a^b y(t)dt\,\hat{j} + \int_a^b z(t)dt\,\hat{k}$$

14.) Arc length function:

$$s(t) = \int_a^t |\bar{r}'(t)|dt = \int_a^t |\bar{T}(t)|dt$$, therefore $\frac{ds(t)}{dt} = |\bar{r}'(t)| = |\bar{T}(t)|$ or $ds = |\bar{r}'(t)|dt = |\bar{T}(t)|dt$

15.) $|\bar{r}'(s)| = |T'(s)| = 1$, therefore $\bar{T}(s) = \hat{T}$

16.) Differential notation: For $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$,

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ and $d\bar{r} = \bar{r}'(t)dt = \bar{T}(t)dt$, therefore $ds = |d\bar{r}|$

17.) Curvature- measures how sharply a curve bends

$$\kappa = \left|\hat{T}'(s)\right| = \left|\frac{d\hat{T}}{ds}\right| = |\bar{r}''(s)| = \frac{|\hat{T}'(t)|}{|\bar{r}'(t)|} = \frac{|\bar{r}'(t) \times \bar{r}''(t)|}{|\bar{r}'(t)|^3} = \frac{\bar{a} \bullet \hat{N}}{|\bar{v}|^2} = \frac{|f''(x)|}{\left[1 + [f'(x)]^2\right]^{3/2}} = \frac{|x'y'' - y'x''|}{\left[(x')^2 + (y')^2\right]^{3/2}}$$

18.) Curvature of a circle of radius a: $\kappa = \frac{1}{a}$

19.) Curvature of a line: $\kappa = 0$

20.) TNB Frame:

$\hat{T}(t) = \frac{\bar{T}(t)}{|\bar{T}(t)|} = \frac{\bar{r}'(t)}{|\bar{r}'(t)|}$ Unit tangent vector

$\hat{N}(t) = \frac{\hat{T}'(t)}{|\hat{T}'(t)|}$ Unit normal vector

$\hat{B}(t) = \hat{T}(t) \times \hat{N}(t)$ Unit binormal vector **(order is important)**

21.) Normal Plane contains $\hat{N}(t)$ and $\hat{B}(t)$

22.) Osculating Plane contains $\hat{T}(t)$ and $\hat{N}(t)$ and osculating circle (circle of curvature at the point of tangency on the curve)

23.) Rectifying plane contains $\hat{T}(t)$ and $\hat{B}(t)$

24.) Velocity in space: $\bar{v}(t) = \dfrac{d\bar{r}}{dt} = \bar{r}'(t) = \dot{\bar{r}}(t)$

25.) Speed in space: $v(t) = |\bar{v}(t)| = \left|\dfrac{d\bar{r}}{dt}\right| = |\bar{r}'(t)| = |\dot{\bar{r}}(t)| = \dfrac{ds}{dt}$

26.) Acceleration in space: $\bar{a}(t) = \dfrac{d\bar{v}}{dt} = \bar{v}'(t) = \dot{\bar{v}}(t) = \dfrac{d^2\bar{r}}{dt^2} = \bar{r}''(t) = \ddot{\bar{r}}(t)$

27.) Tangent and normal components of acceleration:

$$\bar{a}(t) = \frac{dv}{dt}\hat{T} + \kappa v^2 \hat{N} = \frac{d^2 s}{dt^2}\hat{T} + \kappa\left(\frac{ds}{dt}\right)^2 \hat{N} = \frac{\bar{r}' \bullet \bar{r}''}{|\bar{r}'|}\hat{T} + \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|}\hat{N} = \frac{1}{v}\langle \bar{v} \bullet \bar{a},\ |\bar{v} \times \bar{a}| \rangle$$

Or

$$\bar{a}(t) = a_T \hat{T} + a_N \hat{N}$$

where

$$a_T = \frac{dv}{dt} = \frac{d^2 s}{dt^2} = \frac{\bar{r}' \bullet \bar{r}''}{|\bar{r}'|} = \frac{\bar{v} \bullet \bar{a}}{v} = \bar{a} \bullet \hat{v}$$

$$a_N = \kappa v^2 = \kappa\left(\frac{ds}{dt}\right)^2 = \frac{|\bar{r}' \times \bar{r}''|}{|\bar{r}'|} = \frac{|\bar{v} \times \bar{a}|}{v} = |\bar{a} \times \hat{v}|$$

And

$$|\bar{a}(t)| = \sqrt{(a_T)^2 + (a_N)^2}$$