The Factorial Symbol

If $n \geq 0$ is an integer, the **factorial symbol** $n!$ is defined as follows:

$$0! = 1 \qquad 1! = 1$$
$$n! = n(n-1)\cdot \ldots \cdot 3\cdot 2\cdot 1 \qquad \text{if } n \geq 2$$

For example, $2! = 2\cdot 1 = 2$, $3! = 3\cdot 2\cdot 1 = 6$, $4! = 4\cdot 3\cdot 2\cdot 1 = 24$, and so on. Table 2 lists the values of $n!$ for $0 \leq n \leq 6$.

The Symbol $\binom{n}{j}$

or choose

We define the symbol $\binom{n}{j}$, read "n taken j at a time," as follows:

If j and n are integers with $0 \leq j \leq n$, the symbol $\binom{n}{j}$ is defined as

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

In Problems 5-16, evaluate each expression:

6. $\binom{7}{3}$ (n, j)

$n = 7$
$j = 3$

$$= \frac{7!}{3!(7-3)!}$$
$$= \frac{7!}{3!\,4!}$$
$$\frac{7\cdot 6\cdot 5\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}{\cancel{3\cdot 2}\cdot 1\cdot \cancel{4\cdot 3\cdot 2\cdot 1}}$$
$$= 35$$

10. $\binom{100}{98}$ (n, j)

$$= \frac{100!}{98!(100-98)!}$$
$$= \frac{100!}{98!\,2!}$$
$$= \frac{100\cdot 99\cdot \cancel{98\cdot 97\cdot \ldots\cdot 3\cdot 2\cdot 1}}{\cancel{98\cdot 97\cdot 96\cdot \ldots\cdot 3\cdot 2\cdot 1}\cdot 2\cdot 1}$$
$$= \frac{\overset{50}{\cancel{100}}\cdot 99}{\cancel{2}\cdot 1}$$
$$= 50\cdot 99 = 4950$$

Using your calculator: $\binom{n}{r} = {}_nC_r$ ←--**Calculator symbol Found:**

Practice:

~~10.~~ 14. $\binom{60}{20} = 4.19 \times 10^{15}$

~~12.~~ 16. $\binom{37}{19} = 1.767 \times 10^{10}$

Binomial Theorem

Let x and a be real numbers. For any positive integer n, we have

$$\boxed{\begin{aligned}(x+a)^n &= \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \cdots + \binom{n}{j}a^j x^{n-j} + \cdots + \binom{n}{n}a^n \\ &= \sum_{j=0}^{n}\binom{n}{j}x^{n-j}a^j\end{aligned}}$$

$\binom{n}{j}$ is called the **binomial coefficient.**

A Formula for Expanding Binomials: The Binomial Theorem

For any positive integer n,

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots + \binom{n}{n}b^n.$$

In Problems 17-28, expand each expression using the Binomial Theorem.

18. $(x-1)^5 = \binom{5}{0}x^5 + \binom{5}{1}x^{5-1}(-1)^1 + \binom{5}{2}x^{5-2}(-1)^2$

$n=5$

$+ \binom{5}{3}x^{5-3}(-1)^3 + \binom{5}{4}x^{5-4}(-1)^4 + \binom{5}{5}x^{5-5}(-1)^5$

$= 1 \cdot x^5 - 5x^4 + 10x^3 - 10x^2 + 5x - 1$

26. $(\sqrt{x}-\sqrt{3})^4 = \binom{4}{0}(\sqrt{x})^4(-\sqrt{3})^0 + \binom{4}{1}(\sqrt{x})^3(-\sqrt{3})^1$

$n=4$

$1 \quad x^2 \cdot 1 \quad + \quad 4 \; x^{\frac{3}{2}} \cdot \sqrt{3}$

$+ \binom{4}{2}(\sqrt{x})^2(-\sqrt{3})^2 + \binom{4}{3}(\sqrt{x})^1(-\sqrt{3})^3$

$+ \quad 6 \cdot x \cdot 3 \quad + \quad 4 \quad x^{1/2} \cdot -3^{3/2}$

$+ \binom{4}{4}(\sqrt{x})^0(-\sqrt{3})^4$

$+ \; 1 \cdot 1 \cdot 9$

$= x^2 \quad 4x\sqrt{x}\cdot\sqrt{3} + 18x \quad 4\cdot\sqrt{x}\cdot 3\sqrt{3} + 9$

$= x^2 - 4x\sqrt{3x} + 18x - 12\sqrt{3x} + 9 \Leftarrow$

Based on the expansion of $(x + a)^n$, the term containing x^j is

$$\boxed{\binom{n}{n-j} a^{n-j} x^j}$$

30. **What is the coefficient of x^3 in the expansion of $(x-3)^{10}$**

$n = 10$

$j = 3$

$$\binom{10}{10-3} (-3)^{10-3} x^3$$

$$\binom{10}{7} (-3)^7 x^3$$

Issue

Power on a

$10 - 3 = 7$

Coefficient $120(-2187)x^3$

$-262440\, x^3$

32. **What is the coefficient of x^3 in the expansion of $(2x+1)^{12}$**

$n = 12 \quad j = 3$

$$\binom{12}{12-3} (1)^{12-3} (2x)^3$$

$$\binom{12}{9} (1)^9 (2x)^3$$

$$220 (1)(8x^3)$$

$$\underline{1760}\, x^3$$

Coefficient

1760

34. **What is the coefficient of x^2 in the expansion of $(2x-3)^9$**

$n = 9$

$j = 2$

$$\binom{9}{9-2} (2x)^2 (-3)^7$$

$$\binom{9}{7} (4x^2)(-2187)$$

$$36(4x^2)(-2187)$$

$$-314928\, x^2$$

Coefficient