THE PRINCIPLE OF LOGARITHMIC EQUALITY

For any logarithm base a, and for $x, y > 0$,

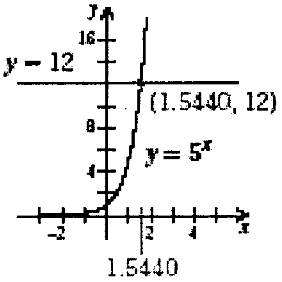
$$\log_a x = \log_a y \qquad x = y.$$

(If the logarithms, base a, of two expressions are the same, then the expressions are the same.)

Because calculators can generally find only common or natural logarithms (without resorting to the change-of-base formula), we usually take the common or natural logarithm on both sides of the equation.

The principle of logarithmic equality is useful anytime a variable appears as an exponent.

EXAMPLE 2 Solve: $5^x = 12$.



$5^x = 12$	
$\log 5^x = \log 12$	Taking the common logarithm on both sides
$x \log 5 = \log 12$	Property 2
$x = \dfrac{\log 12}{\log 5}$	**CAUTION!** This is not $\log \frac{12}{5}$!

This is an exact answer. We cannot simplify further, but we can approximate using a calculator:

$$x = \frac{\log 12}{\log 5} \approx 1.5440.$$

We can also partially check this answer by finding $5^{1.5440}$ using a calculator:

$$5^{1.5440} \approx 12.00078587.$$

We get an answer close to 12, due to the rounding. This checks.

EXAMPLE 3 Solve: $e^{0.06t} = 1500$.

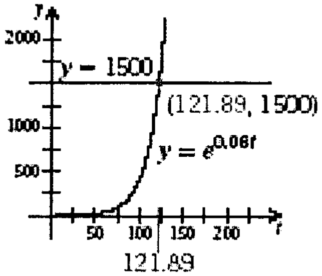
We take the natural logarithm on both sides:

$$
\begin{aligned}
e^{0.06t} &= 1500 \\
\ln e^{0.06t} &= \ln 1500 && \text{Taking ln on both sides} \\
\log_e e^{0.06t} &= \ln 1500 && \text{Definition of natural logarithms} \\
0.06t &= \ln 1500 && \text{Here we use Property 4: } \log_a a^k = k. \\
t &= \frac{\ln 1500}{0.06}.
\end{aligned}
$$

We can approximate using a calculator:

$$t = \frac{\ln 1500}{0.06} \approx \frac{7.3132}{0.06} \approx 121.89.$$

We can also partially check this answer using a calculator.



CHECK:

$$
\begin{array}{r|l}
\multicolumn{2}{c}{e^{0.06t} = 1500} \\
\hline
e^{0.06(121.89)} & ?\ 1500 \\
e^{7.3134} & \\
1500.269444 &
\end{array}
\quad \text{TRUE}
$$

The solution is about 121.89.

Textbook Exercises:

20. $10^x = 8.07$

$$\log 10^x = \log 8.07$$

$$x \cancel{\log 10}^{1} = \log 8.07$$

$$\boxed{x = \log 8.07}$$

22. $e^x = 0.83$

$$\ln(e^x) = \ln(.83)$$

$$x \cancel{\ln e}^{1} = \ln(.83)$$

$$\boxed{x = \ln(.83)}$$

24. $19^x = 143$

$$\log 19^x = \log(143)$$

$$x \log 19 = \log(143)$$

$$\boxed{x = \frac{\log(143)}{\log(9)}}$$

26. $9e^x = 99$

$$e^x = 11$$

$$\ln e^x = \ln 11$$

$$x \cancel{\ln e}^{1} = \ln 11$$

$$\boxed{x = \ln 11}$$

28. $4e^{7x} = 10{,}273$

$$e^{7x} = \frac{10273}{4}$$

$$e^{7x} = 2568.25$$

$$\ln e^{7x} = \ln(2568.25)$$

$$7x \cancel{\ln e}^{1} = \ln(2568.25)$$

$$\boxed{x = \frac{\ln(2568.25)}{7}}$$

34. $135 - (4.7)^x = 0$

$$135 = 4.7^x$$

$$4.7^x = 135$$

$$\log 4.7^x = \log 135$$

$$x \log 4.7 = \log 135$$

$$\boxed{x = \frac{\log 135}{\log 4.7}}$$

Using the properties of logs to solve logarithmic equations:

EXAMPLE 6 Solve: $\log x + \log(x - 3) = 1$.

Here we have common logarithms. It will help us follow the solution to first write in the 10's before we obtain a single logarithmic expression on the left.

$\log_{10} x + \log_{10}(x - 3) = 1$

$\log_{10}[x(x - 3)] = 1$ Using Property 1 to obtain a single logarithm

$x(x - 3) = 10^1$ Writing an equivalent exponential expression

$x^2 - 3x = 10$

$x^2 - 3x - 10 = 0$

$(x + 2)(x - 5) = 0$ Factoring

$x + 2 = 0$ *or* $x - 5 = 0$ Using the principle of zero products

$x = -2$ *or* $x = 5$

CHECK: For -2:

$\log x + \log(x - 3) \stackrel{?}{=} 1$

$\log(-2) + \log(-2 - 3) \ ? \ 1$

The number -2 does *not* check because negative numbers do not have logarithms.

For 5:

$\log x + \log(x - 3) = 1$

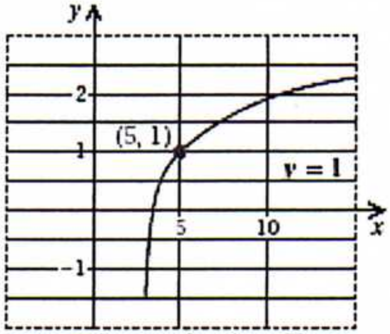
$\log 5 + \log(5 - 3) \ ? \ 1$

$\log 5 + \log 2$

$\log(5 \cdot 2)$

$\log 10$

1 TRUE



$y = \log x + \log(x - 3)$

The solution is 5.

56. $\log_6(x+5)+\log_6 x=2$

$\log_6 x(x+5)=2$

$x(x+5)=6^2$

$x^2+5x=36$

$x^2+5x-36=0$

$(x-4)(x+9)=0$

$\boxed{x=4}$ $x=-9$

(won't work
argument > 0
$x+5$
$-9+5<0$

58. $\log_2(x-1)+\log_2(x+1)=3$

$\log_2[(x-1)(x+1)]=3$

$(x-1)(x+1)=2^3$

$x^2-1=8$

$x^2-9=0$

$(x-3)(x+3)=0$

$\boxed{x=3}$ $\cancel{x=-3}$

← check!

60. $\log_4(x+2)-\log_4(x-1)=1$

$\log_4\left[\frac{x+2}{x-1}\right]=1$

$\frac{x+2}{x-1}=\frac{4^1}{1}$

$x+2=4(x-1)$

$x+2=4x-4$

$-3x=-6$

$\boxed{x=2}$

62. $\log\ (2x-1)-\log\ x=2$

$\log_{10}\left[\frac{2x-1}{x}\right]=2$

$\frac{2x-1}{x}=10^2$

$\frac{2x-1}{x}=100$

$2x-1=100x$

$-1=98x$

$x=\cancel{\frac{-1}{98}}$

no solution