Mat 241 Homework Set 7key – Due ______________

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<u>Directions:</u> **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

<u>Mechanics:</u>

#1. Consider the function defined by $f(x,y)=\dfrac{9x}{x^2+y^2+1}$**.**

A. Determine all critical/stationary points.

$$f(x,y)=\frac{9x}{x^2+y^2+1}$$

$$f_x=\frac{(x^2+y^2+1)9-9x2x}{(x^2+y^2+1)^2}=\frac{9x^2+9y^2+9-18x^2}{(x^2+y^2+1)^2}=\frac{-9x^2+9y^2+9}{(x^2+y^2+1)^2}$$

$$f_x=0\Leftrightarrow\frac{-9x^2+9y^2+9}{(x^2+y^2+1)^2}=0\Rightarrow-9x^2+9y^2+9=0\Leftrightarrow x^2-y^2=1$$

$$f_y=\frac{(x^2+y^2+1)0-9x2y}{(x^2+y^2+1)^2}=\frac{-18xy}{(x^2+y^2+1)^2}$$

$$f_y=0\Leftrightarrow\frac{-18xy}{(x^2+y^2+1)^2}\Rightarrow-18xy=0$$

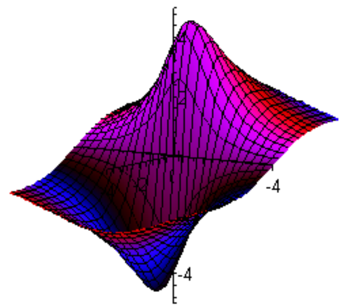
$\therefore x=0, y=0$ **using these in** $f_x=0$**, we get the following**

note;if both x **&** y **=** 0**, then** $x^2-y^2=1$ **does not work. If** $x=0$**,**

then $x^2-y^2=1$ **yields non-real solutions for y. Thus, y can be zero but x cannot.**

critical points $(1,0),(-1,0)$

B. Classify each point as a relative maximum, relative minimum, or a saddle point. Justify your responses.



$$f_x = \frac{-9x^2+9y^2+9}{\left(x^2+y^2+1\right)^2}; f_y = \frac{-18xy}{\left(x^2+y^2+1\right)^2}$$

$$f_{xx} = \frac{\left(x^2+y^2+1\right)^2(-18x)-\left(-9x^2+9y^2+9\right)2\left(x^2+y^2+1\right)2x}{\left(x^2+y^2+1\right)^4}$$

$$f_{yy} = \frac{\left(x^2+y^2+1\right)^2(-18x)-(-18xy)2\left(x^2+y^2+1\right)2y}{\left(x^2+y^2+1\right)^4}$$

$$f_{xy} = \frac{\left(x^2+y^2+1\right)^2(18y)-\left(-9x^2+9y^2+9\right)2\left(x^2+y^2+1\right)2y}{\left(x^2+y^2+1\right)^4}$$

Test for $(1,0)$

$$f_{xx}(1,0) = \frac{(2)^2(-18)-(0)4(2)}{(2)^4} = -4.5 < 0$$

$$f_{yy}(1,0) = \frac{(2)^2(-18)-(0)4(0)}{(2)^4} = -4.5$$

$$f_{xy}(1,0) = \frac{(2)^2(0)-(-9+9)2(1+0+1)2(0)}{(2)^4} = 0$$

$$D(1,0) = f_{xx}(1,0)f_{yy}(1,0)-\left[f_{xy}(1,0)\right]^2 = (-4.5)(-4.5)-0^2 = 20.25$$

$\therefore$ **Since** $f_{xx} < 0$ **and** $D > 0$, $(1,0)$ **is a local maximum.**

Test for $(-1,0)$

$$f_{xx}(-1,0)=\frac{(2)^2(18)-(0)4(-2)}{(2)^4}=4.5<0$$

$$f_{yy}(-1,0)=\frac{(2)^2(18)-(0)4(0)}{(2)^4}=4.5$$

$$f_{xy}(-1,0)=\frac{(2)^2(0)-(-9+9)2(1+0+1)2(0)}{(2)^4}=0$$

$$D(-1,0)=f_{xx}(-1,0)f_{yy}(-1,0)-\left[f_{xy}(-1,0)\right]^2=(4.5)(4.5)-0^2=20.25$$

$\therefore$ Since $f_{xx}>0$ and $D>0$, $(-1,0)$ is a local minimum.

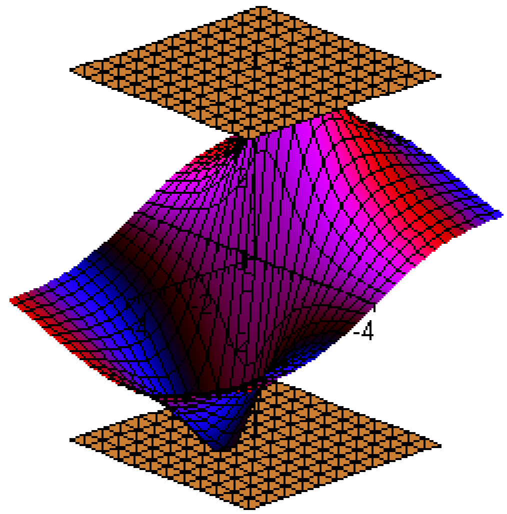
C. Find the equation of the tangent plane for each relative maximum/minimum.

$$f(1,0)=\frac{9}{2};P_1=\left(1,0,\frac{9}{2}\right)$$

$$f(-1,0)=-\frac{9}{2};P_1=\left(1,0,-\frac{9}{2}\right)$$

$$Plane_{\left(1,0,\frac{9}{2}\right)}:z=\frac{9}{2}\,Plane_{\left(1,0,-\frac{9}{2}\right)}:z=-\frac{9}{2}$$

D. Plot the function and any horizontal tangent planes using a suitable 3-D grapher.

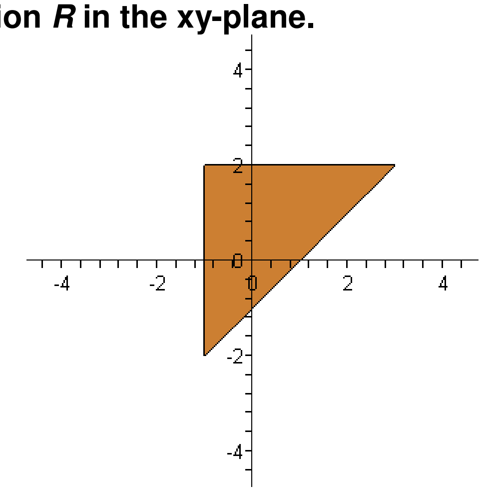


#2. Let *R* be the triangular region in the xy-plane with vertices (-1, -2), (-1, 2) and (3, 2). A metal plate in the shape of *R* is heated so that the temperature at (x, y) is given by:

$$T(x,y) = 2x^2 - xy + y^2 - 2y + 1$$

in degrees Celsius.

A. Sketch the region *R* in the xy-plane.



B. Determine all critical points *within* the region *R*.

$$T(x,y) = 2x^2 - xy + y^2 - 2y + 1$$
$$T_x = 4x - y; \qquad T_{xx} = 4$$
$$T_y = -x + 2y - 2; \qquad T_{yy} = 2$$
$$T_{xy} = -1$$

From $T_x = 0$ **&** $T_y = 0$, **we have critical point** $\left(\frac{2}{7}, \frac{8}{7}\right)$.

$$\therefore T\left(\frac{2}{7}, \frac{8}{7}\right) = -\frac{1}{7}$$

C. Determine all critical points on the boundary.
The boundary is made up of three lines:

x = -1, y = 2, and y = x – 1.

We the function for each line:

$$T(-1,y) = 2 + y + y^2 - 2y + 1 = y^2 - y + 3$$
$$T'(y) = 2y - 1; T'(y) = 0 \Rightarrow y = \frac{1}{2} \text{ and } T''(y) = 2$$

$\therefore \left(-1, \frac{1}{2}\right)$ **is a local minimum. Furthermore,**

$$T(-1,-2) = 1; T(-1,2) = 5;$$

$$T(x,2) = 2x^2 - 2x + 1$$
$$T'(x) = 4x - 2; T'(x) = 0 \Rightarrow x = \frac{1}{2} \text{ and } T''(x) = 4$$

$\therefore \left(\frac{1}{2}, 2\right)$ **is a local minimum. Furthermore,**

$$T(-1,2) = 1; T(3,2) = 13;$$

$$T(x, x-1) = 2x^2 - x(x-1) + (x-1)^2 - 2(x-1) + 1 = 2x^2 - 3x + 4$$

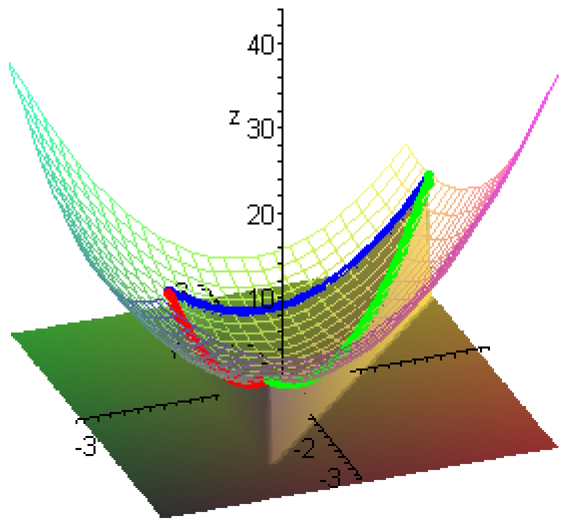
$$T_x(x) = 4x - 3 \Rightarrow T'(x) = 0 \Rightarrow x = \frac{3}{4} \text{ and } T''(x) = 4$$

$\therefore \left(\frac{3}{4}, \frac{-1}{4}\right)$ **is a local minimum. Furthermore,**

$$T(-1,-2) = 5; T(3,2) = 13;$$

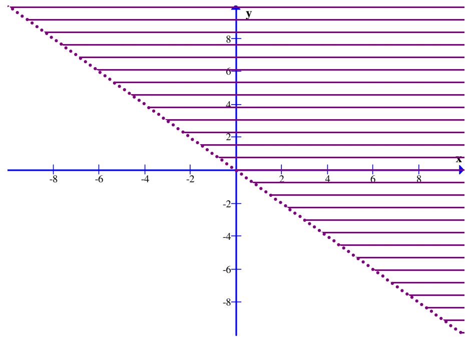
D. At what point in *R* or on its boundary is the temperature maximized? At what point is the temperature minimized? What are the extreme temperatures?

The surfaces has its maximum temperature on the boundary point (3, 2) and its minimum temperature occurring within the region at $\left(\frac{2}{7}, \frac{8}{7}\right)$. The absolute maximum and minimum are 13 and -1/7 respectively.



#3. Consider the function $f(x, y) = x - y^2 - \ln(x + y)$.

A. Sketch the function's domain in R^2.



B. Determine all critical point(s) over its domain.

$$f(x,y) = x - y^2 - \ln(x+y)$$

$$f_x = 1 - \frac{1}{x+y}; f_y = -2y - \frac{1}{x+y}$$

$$f_x = 0 \Leftrightarrow 1 - \frac{1}{x+y} = 0 \Leftrightarrow x + y = 1$$

$$f_y = 0 \Leftrightarrow -2y(x+y) - 1 = 0 \Leftrightarrow -2y - 1 = 0$$

$$\therefore y = \frac{-1}{2}, x = \frac{3}{2}$$

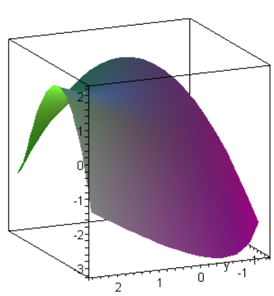
C. Classify the critical point(s) found in part B.

$$f(x,y) = x - y^2 - \ln(x+y)$$

$$f_{xx} = \frac{1}{\left(\frac{3}{2} - \frac{1}{2}\right)^2} = 1; f_{xy} = \frac{1}{\left(\frac{3}{2} - \frac{1}{2}\right)^2} = 1, f_{yy} = -2 + \frac{1}{\left(\frac{3}{2} - \frac{1}{2}\right)^2} = -1;$$

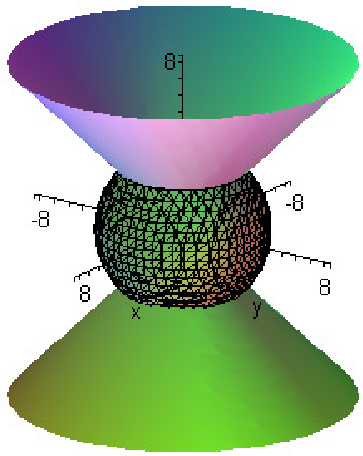
$$D = f_{xx} f_{yy} - \left[f_{xy}\right]^2 = 1(-1) - 1^2 = -2$$

$\therefore$ *Since* $f_{xx} > 0$ **and** $D < 0, \left(\frac{3}{2}, \frac{-1}{2}\right)$ **is a saddle point.**



Conceptual Development

#4. Two surfaces F(x,y,z) = 0 & G(x,y,z) = 0 are said to be orthogonal at a point P if ∇F & ∇G are nonzero at P and the normal lines to the surfaces are perpendicular at P. From this it can be shown that:
"Two surfaces are orthogonal at a point P iff $F_xG_x + F_yG_y + F_zG_z = 0$"
Use this result to show that the sphere $x^2 + y^2 + z^2 = 8$ and the cone $z^2 = x^2 + y^2$ are orthogonal at the point (0, 2, 2).



$$F(x,y,z) = x^2 + y^2 + z^2 - 8; \nabla F(x,y,z) = \langle 2x, 2y, 2z \rangle$$
$$\nabla F(0,2,2) = \langle 0,4,4 \rangle$$
$$G(x,y,z) = x^2 + y^2 - z^2; \nabla G(x,y,z) = \langle 2x, 2y, -2z \rangle$$
$$\nabla G(0,2,2) = \langle 0,4,-4 \rangle$$
$$\nabla F(0,4,4) \cdot \nabla G(0,4,-4) = F_xG_x + F_yG_y + F_zG_z = 0 + 16 - 16 = 0$$

#5. In Calculus I if f is a continuous function of one variable, say for example, $f(x)$, with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself by drawing some pictures in R^2). The purpose of this problem is to show you that this does not extend to functions of two variables. Show that $f(x,y) = 4x^2e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points! (*Based on the article, "Two Mountains Without a Valley", the Mathematics Magazine, Vol.60 No/1 February 1987, p.48)*)

$$f(x,y)=4x^2e^y-2x^4-e^{4y}$$

$$f_x(x,y)=8xe^y-8x^3 : f_{xx}(x,y)=8e^y-24x^2 : f_{xy}(x,y)=8xe^y$$

$$f_y(x,y)=4x^2e^y-4e^{4y} : f_{yy}(x,y)=4x^2e^y-16e^{4y}$$

$$f_x=0 \Rightarrow 8xe^y-8x^3=0 \Leftrightarrow x\left(e^y-x^2\right)=0 \quad \text{/* (1)}$$

$$f_y=0 \Rightarrow 4x^2e^y-4e^{4y}=0 \Leftrightarrow x^2e^y-e^{4y}=0 \quad \text{/* (2)}$$

From (1), x = implies that e^{4y} must be zero which is not possible, hence, $e^y-x^2=0$ ($e^y=x^2$). Substituting in (2):

$$x^4-x^8=0 \Leftrightarrow x^4\left(1-x^4\right)=x^4\left(1-x^2\right)\left(1+x^2\right)=0$$

$$\therefore x=\pm 1, y=0$$

***So*, we have the critical points $(1,0)$ & $(-1,0)$.**

Now, we classify them:

$$f_{xx}(1,0)=8-24=-16 : f_{xy}(1,0)=8 : f_{yy}(1,0)=4-16=-12$$

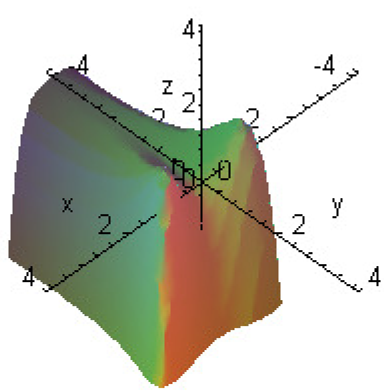
$$D(1,0)=(-16)(-12)-8^2=128$$

$\therefore$ ***Since*, $f_{xx}(1,0)<0$ & $D(1,0)>0$, $(1,0)$ is a relative maximum.**

$$f_{xx}(-1,0)=8-24=-16 : f_{xy}(-1,0)=8 : f_{yy}(-1,0)=4-16=-12$$

$$D(-1,0)=(-16)(-12)-8^2=128$$

$\therefore$ ***Since*, $f_{xx}(-1,0)<0$ & $D(-1,0)>0$, $(-1,0)$ is a relative maximum.**



#6. Use Lagrange multipliers to determine the dimensions of a rectangular box, having an open-top with volume of 32 ft^3, and requiring the least amount of material for its construction.

$$SA(x,y,z)_{\text{minimize}} = xy + 2xz + 2yz$$

$$32 = xyz \to \textbf{constraint } G(x,y,z) = xyz - 32$$

$$\nabla SA = \langle y + 2z, x + 2z, 2x + 2y \rangle : \lambda \nabla G = \langle yz\lambda, xz\lambda, xy\lambda \rangle$$

$$\nabla SA = \lambda \nabla G$$

$$y + 2z = yz\lambda; x + 2z = xz\lambda; 2x + 2y = xy\lambda$$

$$\lambda = \frac{1}{z} + \frac{2}{y} \ \& \ \lambda = \frac{1}{z} + \frac{2}{x} \Rightarrow x = y$$

$$\lambda = \frac{1}{z} + \frac{2}{x} \ \& \ \lambda = \frac{2}{y} + \frac{2}{x} \Rightarrow z = \frac{y}{2} = \frac{x}{2}$$

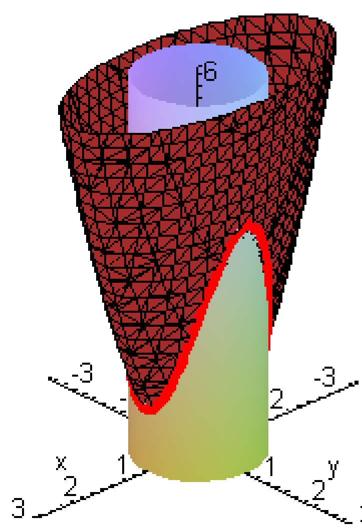
$$\therefore 32 = xyz = x \cdot x \cdot \frac{x}{2} = \frac{x^3}{2}, x^3 = 64 \Rightarrow x = 4$$

$$\therefore x = 4, y = 4, z = 2$$

The box should be $4' \times 4' \times 2'$.

**** So how do we know that this indeed a minimum?***

#7.The figure below shows the intersection of the elliptical paraboloid $z = x^2 + 4y^2$ and the right circular cylinder $x^2 + y^2 = 1$.



A. Use Lagrange multipliers to find the highest and lowest points on the curve of the intersection. That is, find the maxima and minima for $z = x^2 + 4y^2$ subject to the constraint $x^2 + y^2 = 1$.

$$F(x,y) = x^2 + 4y^2, G(x,y) = x^2 + y^2 - 1$$

$$\nabla F(x,y) = \langle 2x, 8y \rangle; \lambda \nabla G(x,y) = \langle 2x\lambda, 2y\lambda \rangle$$

$$\nabla F(x,y) = \lambda \nabla G(x,y)$$

$$2x = 2x\lambda \text{ and } 8y = 2y\lambda$$

$$\text{if } x \neq 0, \text{then } \lambda = 1 \text{ and } y = 0. \text{ Using } x^2 + y^2 = 1$$

$$\text{we get } x^2 = 1 \Rightarrow (1,0),(-1,0)$$

$$\text{if } y \neq 0, \text{then } \lambda = 4 \text{ and } x = 0. \text{ Using } x^2 + y^2 = 1$$

$$\text{we get } y^2 = 1 \Rightarrow (0,1),(0,-1)$$

$$F(1,0) = 1^2 + 4 \cdot 0^2 = 1; F(-1,0) = (-1)^2 + 4 \cdot 0^2 = 1$$

$$F(0,1) = 0^2 + 4 \cdot 1^2 = 4; F(0,-1) = 0^2 + 4 \cdot (-1)^2 = 4$$

$\therefore$ **The maxumum height is 4 and the minimum height is1.**

B. Determine parametric equations for the curve of intersection.

$$x^2 + y^2 = 1 \text{ and } z = x^2 + 4y^2$$

$$\text{let } x(t) = \cos t, y(t) = \sin t, \text{then } z(t) = \cos^2 t + 4\sin^2 t$$

C. What is the domain for *t* in your equations found in part B?

$$D : (t \,|\, t \in \square)$$

D. Using z(t) from part B, what is the maximum height and minimum height it can take on and how does this compare to part A's answer?

$$z(t) = \cos^2 t + 4\sin^2 t$$

$$z'(t) = -2\cos t \sin t + 8\sin t \cos t = 6\sin t \cos t$$

$$z'(t) = 0 : 6\sin t \cos t = 0 \Rightarrow t = k\pi, k = 0, \pm 1, \pm 2, \ldots; t = \frac{\pi}{2} + k\pi, k = 0, \pm 1, \pm 2, \ldots$$

$$\text{When } t = \pi, x(\pi) = -1, y(\pi) = 0 \Rightarrow z = (-1)^2 + 4 \cdot 0 = 1$$

$$\text{When } t = \frac{\pi}{2}, x\left(\frac{\pi}{2}\right) = 0, y\left(\frac{\pi}{2}\right) = 1 \Rightarrow z = (0)^2 + 4 \cdot 1 = 4$$

$\therefore z$ **has a maxumum height of 4 and a minimum height of 1. As expected.**

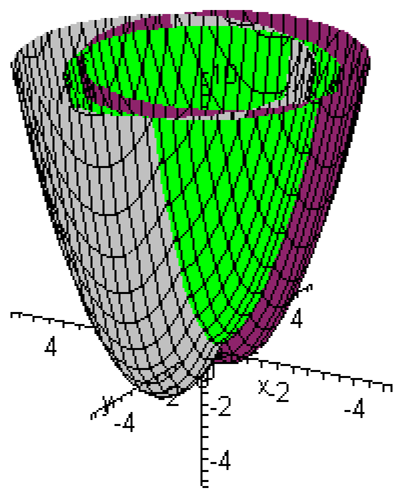
A Challenge for you – I was sitting in my office and started plotting several paraboloids. I plotted:

$$z = x^2 + (y-1)^2 - 2$$

$$z = (x-1)^2 + y^2 - 1$$

$$z = x^2 + y^2$$

The outcome is shown:



I realized that there could very well be a plane that is simultaneously tangent to all three paraboloids. Indeed, there is and we shall find this together.

Step 1. Name the surfaces:

$$S_1(x,y,z): x^2 + y^2 - z = 0$$

$$S_2(x,y,z): (x-1)^2 + y^2 - 1 - z = 0$$

$$S_3(x,y,z): x^2 + (y-1)^2 - 2 - z = 0$$

Suppose the plane we seek touches the surfaces at the points $P_1(x_1,y_1,z_1), P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$ respectively.

Determine the gradient vectors for the three surfaces at the points P_1, P_2, and P_3.

Step 2. The corresponding components from the three gradient vectors are equivalent. Using that fact and the surface equations S_2 & S_3, show that:

$$x_2 = x_1 + 1, y_2 = y_1, z_2 = z_1 - 1$$
$$x_3 = x_1, y_3 = y_1 + 1, z_3 = z_1 - 2$$

Now write P_2 & P_3 in terms of $x_1, y_1, \& z_1$. Could you see how this same result could be deduced just from looking at the equations of $S_1, S_2, and\ S_3$?

Step 3. Since we now have three points on the plane all in terms of $x_1, y_1, \& z_1$, create the two vectors in the plane $P_1P_2\ and\ P_1P_3$.

Step 4. Determine the cross-product of the vectors found in Step 3. This vector is a normal vector to the plane. It is also collinear with $\nabla S_1(x_1, y_1, z_1)$.

Use that fact as well as S_1 to show that $x_1 = -\frac{1}{2}, y_1 = -1, and\ z_1 = \frac{5}{4}$

Step 5. Using the information from step 4 show that the equation of the tangent plane which is tangent to all three paraboloids is given by:

$$4x + 8y + 4z + 5 = 0$$