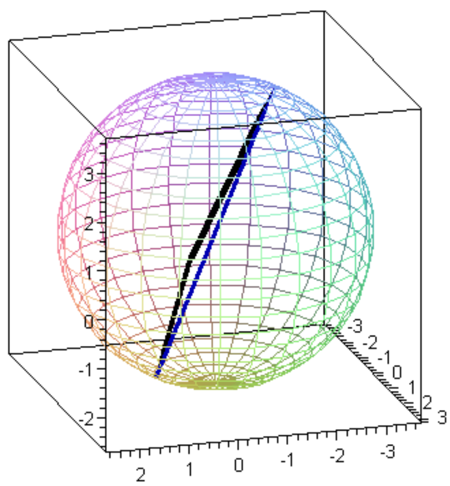
Mat 241 Chapter 12 Test – Key Fall 2008

#1. A $d=\sqrt{\left(1-(-2)\right)^2+\left(1-(-1)\right)^2+\left(-2-3\right)^2}=\sqrt{9+4+25}=\sqrt{38}$

B. $Mid=\left(\frac{-2+1}{2},\frac{-1+1}{2},\frac{3+(-2)}{2}\right)=\left(\frac{-1}{2},0,\frac{1}{2}\right)$

$Sphere=\left(x-h\right)^2+\left(x-h\right)^2+\left(x-h\right)^2=r^2$

C. $\left(x+\frac{1}{2}\right)^2+y^2+\left(z-\frac{1}{2}\right)^2=\frac{19}{2}$



D. $\overrightarrow{PQ}=\left\langle 1-(-2),1-(-1),-2-3\right\rangle=\left\langle 3,2,-5\right\rangle$

E.

$$\vec{U}_{\overrightarrow{pq}}=\frac{\overrightarrow{PQ}}{\left\|\overrightarrow{PQ}\right\|}=\frac{1}{\sqrt{3^2+2^2+(-5)^2}}\left\langle 3,2,-5\right\rangle$$

$$=\frac{1}{\sqrt{38}}\left\langle 3,2,-5\right\rangle=\left\langle\frac{3}{\sqrt{38}},\frac{2}{\sqrt{38}},\frac{-5}{\sqrt{38}}\right\rangle or \left\langle\frac{3\sqrt{38}}{38},\frac{\sqrt{38}}{19},\frac{-5\sqrt{38}}{38}\right\rangle$$

#2.

A. $\theta=\cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\right)=\cos^{-1}\left(\frac{-4-1+4}{\sqrt{6}\sqrt{21}}\right)\approx 95.11°$

$$\alpha = \cos^{-1}\left(\frac{a \cdot \vec{i}}{\|\vec{w}\|}\right) \to \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66°$$

B. $$\beta = \cos^{-1}\left(\frac{b \cdot \vec{j}}{\|\vec{w}\|}\right) \to \cos^{-1}\left(\frac{-1}{\sqrt{6}}\right) \approx 114°$$

$$\gamma = \cos^{-1}\left(\frac{c \cdot \vec{k}}{\|\vec{w}\|}\right) \to \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35°$$

C. $$proj_{\vec{v}}\vec{u} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2}\vec{v} = \frac{-1}{21}\langle -4,1,2 \rangle = \left\langle \frac{4}{21}, \frac{-1}{21}, \frac{-2}{21} \right\rangle$$

$$orth_{\vec{v}}\vec{u} = \vec{u} - proj_{\vec{v}}\vec{u} = \langle 1,-1,2 \rangle - \left\langle \frac{4}{21}, \frac{-1}{21}, \frac{-2}{21} \right\rangle = \left\langle \frac{17}{21}, \frac{-20}{21}, \frac{44}{21} \right\rangle$$

D. $$\vec{w} \times \vec{z} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 0 & -1 \end{vmatrix} = (1-0)\vec{i} - (-1-4)\vec{j} + (0-(-2))\vec{k}$$

$$= \vec{i} + 5\vec{j} + 2\vec{k} = \langle 1,5,2 \rangle$$

E. $$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} 1 & -1 & 2 \\ -4 & 1 & 2 \\ 1 & -1 & 2 \end{vmatrix} = 1(2-(-2)) - (-1)(-8-2) + 2(4-1)$$

$$= 4 - 10 + 6 = 0 \therefore \textbf{They are coplanar}$$

#3.

$$A(3,-1,1), B(4,0,2), C(6,3,1)$$

$$\overline{AB} = \langle 4-3, 0-(-1), 2-1 \rangle = \langle 1,1,1 \rangle$$

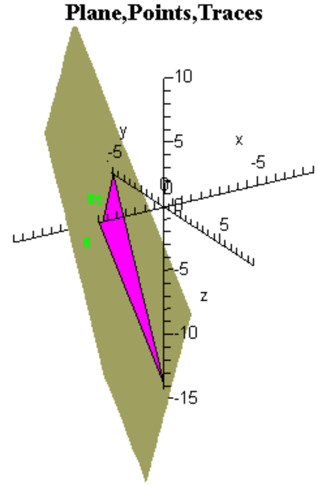
$$\overline{AC} = \langle 6-3, 3-(-1), 1-1 \rangle = \langle 3,4,0 \rangle$$

$$\vec{n} = \overline{AB} \times \overline{AC} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 3 & 4 & 0 \end{vmatrix} = -4\vec{i} + 3\vec{j} + 1\vec{k}$$

$$-4x + 3y + z = 3 \cdot (-4) + (-1) \cdot 3 + 1 \cdot 1 = -14$$

$$\therefore \textit{plane's equation}: 4x - 3y - z = 14$$

Plane,Points,Traces



#4.

$$\vec{v} = \langle 1-4, 1-(-1), 5-2 \rangle = \langle -3, 2, 3 \rangle$$

$$x = 4 - 3t, y = -1 + 2t, z = 2 + 3t \quad \text{/* parametric equations}$$

$$\frac{x-4}{-3} = \frac{y+1}{2} = \frac{z-2}{3} \quad \text{/* symmetric equation}$$

#5. A.

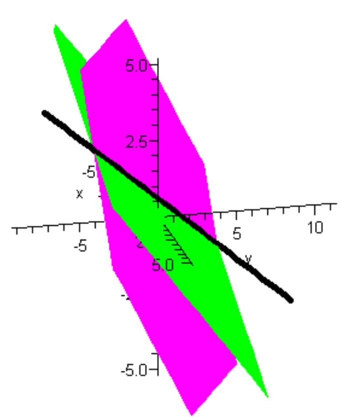
$$\overrightarrow{n_1} = \langle 1, -2, 1 \rangle ; \overrightarrow{n_2} = \langle 2, 3, -2 \rangle ; \left\| \overrightarrow{n_1} \right\| = \sqrt{3} ; \left\| \overrightarrow{n_2} \right\| = \sqrt{14}$$

$$\theta = \cos^{-1}\left(\frac{\left| \overrightarrow{n_1} \cdot \overrightarrow{n_2} \right|}{\left\| \overrightarrow{n_1} \right\| \left\| \overrightarrow{n_2} \right\|} \right) = \cos^{-1}\left(\frac{6}{\sqrt{42}} \right) \approx 22.21°$$

** We define the angle between two planes to be the acute angle.

B.

Intersecting Planes



$$\overrightarrow{n_1} = \langle 1,-2,1 \rangle ; \overrightarrow{n_2} = \langle 2,3,-2 \rangle$$

$$\overrightarrow{n_1} \times \overrightarrow{n_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{vmatrix} = (4-3)\vec{i} - (-2-2)\vec{j} + (3-(-4))\vec{k} = \vec{i} + 4\vec{j} + 7\vec{k}$$

$$\begin{aligned} x + y + z &= 0 \\ x + 2y + 3z &= 1 \end{aligned}$$

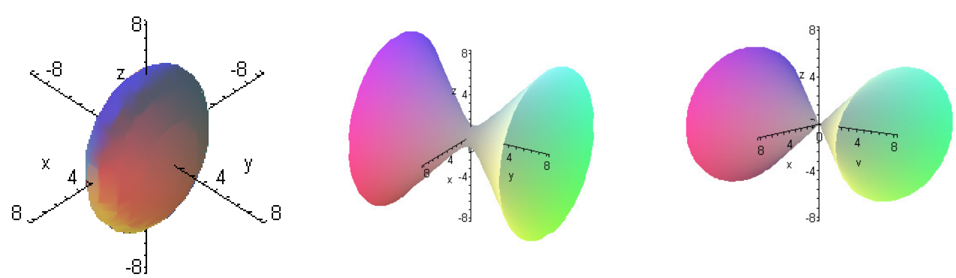
$$\text{let } z = t$$

$$\begin{aligned} x + y + t &= 0 \\ x + 2y + 3t &= 1 \end{aligned} \Rightarrow \begin{aligned} x + y &= -t \\ x + 2y &= 1 - 3t \end{aligned} \Rightarrow y = 1 - 2t$$

$$\therefore x = t - 1, y = 1 - 2t, z = t \therefore \textit{Answer is not unique}$$

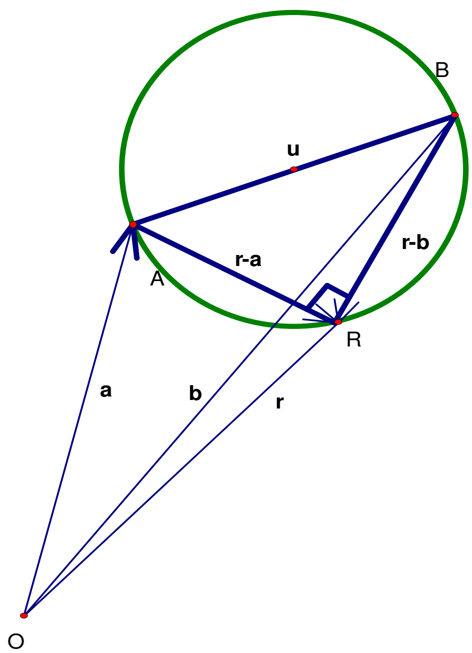
#6. $\|\tau\| = \|r\|\|F\|\sin\theta \rightarrow (0.4m)(50N)\sin 120^{\circ} \approx 17.32N \cdot m = 17.32J$

#7.



#8.

There is an easy way and a more involved way. Let's look at the easy way first. Consider the circle shown below:



Since

$\left(\vec{r}-\vec{a}\right)\cdot\left(\vec{r}-\vec{b}\right)=0 \Rightarrow$ they are perpendicular and thus the vector connecting their two tails passes through the center of the circle that contains points A, B, and R.

The center of the circle is the midpoint of that diamter $center=\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2},\frac{a_3+b_3}{2}\right)$.

The radius of the sphere is equal to one-half the length of the diameter. The diameter's length is as follows:

$\vec{a}+\vec{u}=\vec{b} \Leftrightarrow \vec{u}=\vec{b}-\vec{a}$

$so: radius=\frac{1}{2}\left\|\vec{u}\right\|=\frac{1}{2}\left\|\vec{b}-\vec{a}\right\|=\frac{1}{2}\sqrt{\left(b_1-a_1\right)^2+\left(b_2-a_2\right)^2+\left(b_3-a_3\right)^2}$

$\therefore The\ equation\ of\ the\ sphere\ is:$

$$\left(x-\frac{a_1+b_1}{2}\right)^2+\left(y-\frac{a_2+b_2}{2}\right)^2+\left(z-\frac{a_3+b_3}{2}\right)^2=\frac{\left(b_1-a_1\right)^2+\left(b_2-a_2\right)^2+\left(b_3-a_3\right)^2}{4}$$

We can also just crank it out. Enjoy1

$$(\vec{r}-\vec{a})\cdot(\vec{r}-\vec{b})=(\langle x,y,z\rangle-\langle a_1,a_2,a_3\rangle)\cdot(\langle x,y,z\rangle-\langle b_1,b_2,b_3\rangle)$$
$$=\langle x-a_1,y-a_2,z-a_3\rangle\cdot\langle x-b_1,y-b_2,z-b_3\rangle=$$
$$(x-a_1)(x-b_1)+(y-a_2)(y-b_2)+(z-a_3)(z-b_3)$$
$$=x^2-b_1x-a_1x+a_1b_1+y^2-b_2y-a_2y+a_2b_2+z^2-b_3z-a_3z+a_3b_3$$
$$=x^2-(b_1+a_1)x+y^2-(b_2+a_2)y+z^2-(b_3+a_3)z+a_1b_1+a_2b_2+a_3b_3=0$$
$$\Leftrightarrow$$
$$x^2-(b_1+a_1)x+y^2-(b_2+a_2)y+z^2-(b_3+a_3)z=-a_1b_1-a_2b_2-a_3b_3$$
$$\left(x-\frac{b_1+a_1}{2}\right)^2-\frac{(b_1+a_1)^2}{4}+\left(y-\frac{b_2+a_2}{2}\right)^2-\frac{(b_2+a_2)^2}{4}+\left(z-\frac{b_3+a_3}{2}\right)^2-\frac{(b_3+a_3)^2}{4}$$
$$=-a_1b_1-a_2b_2-a_3b_3$$
$$\Leftrightarrow$$
$$\left(x-\frac{b_1+a_1}{2}\right)^2+\left(y-\frac{b_2+a_2}{2}\right)^2+\left(z-\frac{b_3+a_3}{2}\right)^2=\frac{(b_1+a_1)^2}{4}+\frac{(b_2+a_2)^2}{4}+\frac{(b_3+a_3)^2}{4}-a_1b_1-a_2b_2-a_3b_3$$
$$\Leftrightarrow$$
$$\left(x-\frac{b_1+a_1}{2}\right)^2+\left(y-\frac{b_2+a_2}{2}\right)^2+\left(z-\frac{b_3+a_3}{2}\right)^2=$$
$$\frac{b_1^2+2a_1b_1+a_1^2+b_2^2+2a_2b_2+a_2^2+b_3^2+2a_3b_3+a_3^2-4a_1b_1-4a_2b_2-4a_3b_3}{4}$$
$$=\frac{b_1^2-2a_1b_1+a_1^2+b_2^2-2a_2b_2+a_2^2+b_3^2-2a_3b_3+a_3^2}{4}$$
$$=\frac{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}{4}$$
$$\therefore Sphere\ Eq:\left(x-\frac{b_1+a_1}{2}\right)^2+\left(y-\frac{b_2+a_2}{2}\right)^2+\left(z-\frac{b_3+a_3}{2}\right)^2=\frac{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}{4}$$
$$\therefore Center:\left(\frac{b_1+a_1}{2},\frac{b_2+a_2}{2},\frac{b_3+a_3}{2}\right)$$
$$\therefore Radius:\sqrt{\frac{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}{4}}=\frac{1}{2}\sqrt{(b_1-a_1)^2+(b_2-a_2)^2+(b_3-a_3)^2}$$

Trust your instincts!