Mat 241 Homework Set 8 – Due ______________

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Directions: Show all algebraic steps neatly and concisely using proper mathematical symbolism. When graphs and technology are to be implemented, do so appropriately.

Mechanics:

For the following two integrals do the following:

A. Sketch the region of integration for the two integrals shown.
B. Compute the integrals in 1 & 2 exactly.
C. Write each integral with the order of integration reversed and then compute each of the “new” integrals.

1.

\[
\int_0^2 \int_0^{\frac{1}{2}} \left( x + y \right) dxdy
\]

\[
= \int_0^2 \left[ \frac{x^2}{2} + yx \right]_{\frac{1}{2}}^{\frac{1}{2}} dy = \int_0^2 \left( \frac{1}{2} + y - \frac{5y^2}{8} \right) dy = \left[ \frac{y}{2} + \frac{y^2}{2} - \frac{5y^3}{24} \right]_0^1 = \frac{4}{3}
\]

\[
\int_0^2 \int_0^{\frac{1}{2}} \left( x + y \right) dxdy = \int_0^{\frac{1}{2}} \int_0^{2x} \left( x + y \right) dydx
\]

\[
= \int_0^1 \left[ xy + \frac{y^2}{2} \right]_{\frac{1}{2}}^{2x} dx = \int_0^1 \left( 4x^2 \right) dx = \left[ \frac{4x^3}{3} \right]_0^1 = \frac{4}{3}
\]
For the following two integrals do the following:

D. Sketch the region of integration.
E. Write each integral with the order of integration reversed.
F. Compute each of the “new” integrals. Hint: On #4 utilize a Taylor series using the first 6 terms to approximate one part of the region.

\[
\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} y \, dx \, dy = \int_{0}^{\frac{\pi}{4}} \int_{0}^{\sqrt{2}} r \sin \theta \, dr \, d\theta
\]

= \int_{0}^{\frac{\pi}{4}} \frac{\sin \theta}{2} \left[ \sqrt{2} \cdot \sin \frac{\theta}{2} \right] d\theta = \frac{\pi}{8} \left(1 - \frac{\pi}{2}\right) = \frac{1}{3}

For the following two integrals do the following:

\[
\int_{0}^{1} \int_{0}^{\sqrt{1-x^2}} y \, dy \, dx = \int_{0}^{1} \frac{\sqrt{1-x^2}}{2} \left[ \frac{x}{2} \right] dx = \frac{1}{2} \left(\frac{1}{3} - \frac{x^3}{3}\right)^{\frac{1}{4}} = \frac{1}{3}
\]

3. \[
\int_{\frac{4}{\ln y}}^{4} \frac{1}{\ln y} \, dy = \int_{e^{\frac{1}{4}}}^{4} \frac{1}{\ln y} \, dy = \int_{1}^{4} \frac{x}{\ln y} \, dy = \int_{2}^{6} 2dy = 6
\]
4.
\[
\begin{align*}
\int_0^1 \int_0^2 \frac{e^x}{\sqrt{x}} \, dy \, dx &= \int_0^1 e^x \, dy \int_0^2 \frac{1}{\sqrt{x}} \, dx = \int_0^1 \frac{e^x}{1} \, dy \int_0^2 \frac{1}{\sqrt{x}} \, dx \\
&= \left[ e^x \right]_1^2 + \left[ \frac{e^x}{\sqrt{x}} \right]_1^2 = e - 1 + \frac{e}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{x^2}{2\sqrt{x}} + \ldots \, dx \\
&= e - 1 + \int_1^2 \left( \frac{1}{\sqrt{x}} + \frac{x^2}{2\sqrt{x}} + \frac{x^3}{6\sqrt{x}} \right) \, dx \\
&\to e - 1 + 3.470 \\
\therefore &\quad e + 2.47
\end{align*}
\]

Concept development and applications.

#5. Use a CAS to compute the following two integrals showing that they are not the same. Why doesn’t this contradict Fubini’s Theorem?

\[
\begin{align*}
\int_0^1 \int_0^1 (x - y) \, dy \, dx &\quad \text{and} \quad \int_0^1 \int_0^1 (x - y) \, dx \, dy \\
\text{The function has an infinite discontinuity at the origin so Fubini’s Theorem does not apply.}
\end{align*}
\]

Can you compute these by hand?(not required)

#6. The order of integration in a double integral is largely a matter of choice but sometimes the order can be the difference between a straightforward evaluation as opposed to a very difficult if not impossible evaluation.

Consider the function \( f(x, y) = x \cos(xy) \) on \( R = \left[ 0, \frac{\pi}{2} \right] \times [0,1] \).
A. Integrate the function over the specified region with respect to y first (i.e. dydx).

\[ f(x, y) = x \cos(xy) \text{ on } R = \left[0, \frac{\pi}{2}\right] \times [0, 1]. \]

\[
\int_0^\frac{\pi}{2} \int_0^1 x \cos(xy) \, dy \, dx = \int_0^\frac{\pi}{2} \left( \sin(xy) \right)_0^1 \, dx = \int_0^\frac{\pi}{2} \sin(x) \, dx = -\cos(x) \bigg|_0^{\frac{\pi}{2}} = 1
\]

B. Integrate the function over the specified region with respect to x first and indicate when the difficulty arises. (i.e. dxdy).

\[ f(x, y) = x \cos(xy) \text{ on } R = \left[0, \frac{\pi}{2}\right] \times [0, 1]. \]

\[
\int_0^1 \int_0^{\frac{\pi}{2}} x \cos(xy) \, dx \, dy = \int_0^1 \left( \frac{x \sin(xy)}{y} \right)_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{\sin(xy)}{y} \, dx \, dy = \int_0^1 \left( \frac{x \sin(xy)}{y} \right)_0^{\frac{\pi}{2}} + \frac{\cos(xy)}{y^2} \bigg|_0^{\frac{\pi}{2}} \, dy \rightarrow
\]

\[
= \int_0^1 \frac{\pi \sin\left(\frac{\pi y}{2}\right)}{y} \, dy + \int_0^1 \left( \frac{\cos\left(\frac{\pi y}{2}\right)}{y^2} - \frac{1}{y^2} \right) \, dy
\]

\[
= \int_a^b \frac{\sin(y)}{y} \, dy \quad \text{/* is not an elementary integral.}
\]
#7. A rectangular plate of sides lengths $a$ and $b$ is subjected to a normal force (that is perpendicular to the plate). The pressure, $p$, at any point on the plate is proportional to the square of the distance of that point from one corner. Find the total force on the plate [Note: Pressure is force per unit area].

\[
F = \int_0^a \int_0^b k \left( x^2 + y^2 \right) dy dx = k \int_0^a \left( x^2 y + \frac{y^3}{3} \right) dx = k \left( \frac{x^3 b}{3} + \frac{b^3}{3} \right) \bigg|_0^a = abk \left( \frac{a^2 + b^2}{3} \right)
\]

#8. Find the volume of the solid bounded by the paraboloid $z = 9x^2 + y^2$ above, by the plane $z = 0$ below, and laterally by the planes $x = 0$, $y = 0$, $x = 3$, and $y = 2$. Sketch the Region in the $xy$ – plane and indicate your directions of integrations.

\[
V = \int_0^3 \int_0^2 \left( 9x^2 + y^2 \right) dx dy = \int_0^3 \left( 3x^3 + y^2 x \right) \bigg|_0^2 dy = \int_0^3 \left( 81 + 3y^2 \right) dy = 170
\]
#9. Find the volume of the solid bounded by the two surfaces. Sketch the Region in the xy – plane and indicate your directions of integrations.

\[ z = x^2 + 3y^2 \text{ & } z = 4 - y^2. \]

\[ f(x, y) = 4 - y^2 - (x^2 + 3y^2) = 4 - x^2 - 4y^2; x^2 + 3y^2 = 4 - y^2 \iff \frac{x^2}{4} + \frac{y^2}{1} = 1 \]

\[ V = 4 \int_0^{2\sqrt{1-y^2}} \int_0^{\sqrt{1-y^2}} (4 - x^2 - 4y^2) dx dy = 4 \int_0^{\sqrt{1-y^2}} \left( 4x - \frac{x^3}{3} - 4y^2x \right) dy \]

\[ = 4 \int_0^{\sqrt{1-y^2}} \left( 8\sqrt{1-y^2} - \frac{8(1-y^2)\sqrt{1-y^2}}{3} - 8y^2\sqrt{1-y^2} \right) dy \]

\[ = \frac{32}{3} \int_0^{\sqrt{1-y^2}} \left( 2\sqrt{1-y^2} - 2y^2\sqrt{1-y^2} \right) dy \]

\[ = \frac{64}{3} \int_0^{\sqrt{1-y^2}} \left( \sqrt{1-y^2}^2 - y^2\sqrt{1-y^2} \right) dy \]

\[ = \frac{64}{3} \int_0^{\sqrt{1-y^2}} (\cos^2 \theta - \sin^2 \theta \cos^2 \theta) d\theta = \frac{64}{3} \int_0^{\sqrt{1-y^2}} \cos^4 \theta d\theta \quad \text{/* } y = \sin \theta \]

\[ \Rightarrow \frac{64}{3} \int_0^{\sqrt{1-y^2}} \cos^4 \theta d\theta = \frac{64}{3} \left( \frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} \right) \bigg|_0^{\sqrt{1-y^2}} \]

\[ = 4\pi \]
#10. Find the volume of the solid formed by the two paraboloids $z = x^2 + 3y^2$ \& $z = 9 - 2x^2 - y^2$. Sketch the Region in the xy – plane and indicate your directions of integrations.

\[ f(x, y) = 9 - 3x^2 - 4y^2; \quad \frac{x^2}{3} + \frac{y^2}{9} = 1 \]

\[ V = \int_0^3 \int_0^{\sqrt[3]{9-4y^2}} \left(9 - 3x^2 - 4y^2\right) \, dx \, dy = \int_0^{\frac{3}{\sqrt{3}}} \left(9x - x^2 - 4y^2\right) \frac{1}{\sqrt[3]{9-4y^2}} \, dy \]

\[ = \int_0^{\frac{3}{\sqrt{3}}} \left(9\sqrt{9-4y^2} - \frac{(9 - 4y^2)\sqrt{9-4y^2}}{3\sqrt{3}} - \frac{4y^2\sqrt{9-4y^2}}{\sqrt{3}}\right) \, dy \]

\[ = \frac{4}{3\sqrt{3}} \int_0^{\frac{3}{\sqrt{3}}} (18\sqrt{9-4y^2} - 8y^2\sqrt{9-4y^2}) \, dy \quad \text{/let } u = 2y \]

\[ = \frac{4}{3\sqrt{3}} \int_0^3 (9\sqrt{9-u^2} - u^2\sqrt{9-u^2}) \, dy \quad \text{/let } u = 3\sin \theta \]

\[ = \frac{4}{3\sqrt{3}} \int_0^{\frac{\pi}{2}} 81\left(\cos^2 \theta - \sin^2 \theta \cos^2 \theta\right) \, d\theta = \frac{108}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta \]

\[ * \int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta = \frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32} + C \]

\[ \Rightarrow \frac{108}{\sqrt{3}} \int_0^{\frac{\pi}{2}} \cos^4 \theta \, d\theta = \frac{108}{\sqrt{3}} \left(\frac{3\theta}{8} + \frac{\sin(2\theta)}{4} + \frac{\sin(4\theta)}{32}\right)_{\frac{\pi}{2}}^0 = \frac{27\sqrt{3}\pi}{4} \]