# Mat 241 Homework Set 7 – Due ______________

***Professor David Schultz***

***Directions:*** **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

## Mechanics:

**#1. Consider the function defined by** $f\left(x,y\right)=\frac{9x}{x^2+y^2+1}$.

A. **Determine all critical/stationary points.**
B. **Classify each point as a relative maximum, relative minimum, or a saddle point. Justify your responses.**
C. **Find the equation of the tangent plane for each relative maximum/minimum.**
D. **Plot the function and any horizontal tangent planes using a suitable 3-D grapher.**

**#2. Let *R* be the triangular region in the xy-plane with vertices (-1, -2), (-1, 2) and (3, 2). A metal plate in the shape of *R* is heated so that the temperature at (x, y) is given by:**

$$T(x,y)=2x^2-xy+y^2-2y+1$$

**in degrees Celsius.**

A. **Sketch the region *R* in the xy-plane.**
B. **Determine all critical points *within* the region *R*.**
C. **Determine all critical points on the boundary.**
D. **At what point in *R* or on its boundary is the temperature maximized? At what point is the temperature minimized? What are the extreme temperatures?**

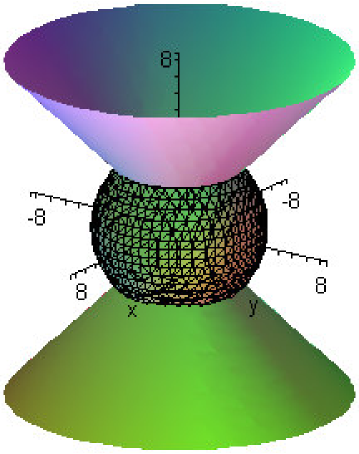
**#3. Consider the function** $f(x,y)=x-y^2-\ln(x+y)$.

A. **Sketch the function's domain in** $\mathrm{R}^2$.
B. **Determine all critical point(s) over its domain.**
C. **Classify the critical point(s) found in part B.**

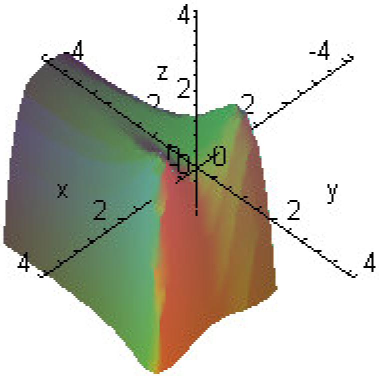
## Conceptual Development

#4. Two surfaces F(x,y,z) = 0 & G(x,y,z) = 0 are said to be orthogonal at a point P if $\nabla F$ & $\nabla G$ are nonzero at P and the normal lines to the surfaces are perpendicular at P. From this it can be shown that:
"Two surfaces are orthogonal at a point P iff $F_x G_x + F_y G_y + F_z G_z = 0$"

Use this result to show that the sphere $x^2 + y^2 + z^2 = 8$ and the cone $z^2 = x^2 + y^2$ are orthogonal at the point (0, 2, 2).

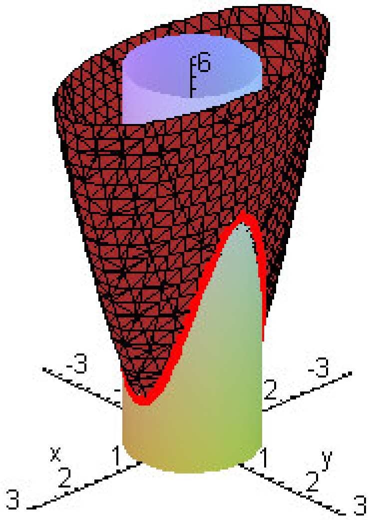


#5. In Calculus I if $f$ is a continuous function of one variable, say for example, $f(x)$, with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself by drawing some pictures in R$^2$). The purpose of this problem is to show you that this does not extend to functions of two variables. Show that $f(x, y) = 4x^2 e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points! (*Based on the article, "Two Mountains Without a Valley", the Mathematics Magazine, Vol.60 No/1 February 1987, p.48)*



**#6. Use Lagrange multipliers to determine the dimensions of a rectangular box, having an open-top with volume of 32 ft$^3$, and requiring the least amount of material for its construction.**

**#7.The figure below shows the intersection of the elliptical paraboloid $z = x^2 + 4y^2$ and the right circular cylinder $x^2 + y^2 = 1$.**



**A. Use Lagrange multipliers to find the highest and lowest points on the curve of the intersection. That is, find the maxima and minima for $z = x^2 + 4y^2$ subject to the constraint $x^2 + y^2 = 1$.**

**B. Determine parametric equations for the curve of intersection.**

**C. What is the domain for *t* in your equations found in part B?**

**D. Using z(t) from part B, what is the maximum height and minimum height it can take on and how does this compare to part A's answer?**

## Some True/False Practice – Handout

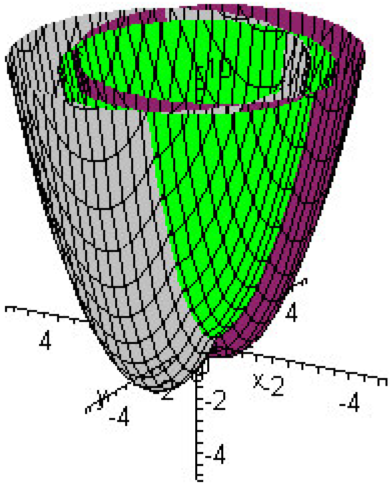
**A Challenge for you – I was sitting in my office and started plotting several paraboloids. I plotted:**

$$z = x^2 + (y-1)^2 - 2$$

$$z = (x-1)^2 + y^2 - 1$$

$$z = x^2 + y^2$$

**The outcome is shown:**



**I realized that there could very well be a plane that is simultaneously tangent to all three paraboloids. Indeed, there is and we shall find this together.**

**Step 1. Name the surfaces:**

$$S_1(x,y,z): x^2 + y^2 - z = 0$$

$$S_2(x,y,z): (x-1)^2 + y^2 - 1 - z = 0$$

$$S_3(x,y,z): x^2 + (y-1)^2 - 2 - z = 0$$

**Suppose the plane we seek touches the surfaces at the points $P_1(x_1,y_1,z_1), P_2(x_2,y_2,z_2)$, and $P_3(x_3,y_3,z_3)$ respectively.**

**Determine the gradient vectors for the three surfaces at the points $P_1, P_2$, and $P_3$.**

**Step 2. The corresponding components from the three gradient vectors are equivalent. Using that fact and the surface equations $S_2$ & $S_3$, show that:**

$$x_2 = x_1 + 1, y_2 = y_1, z_2 = z_1 - 1$$
$$x_3 = x_1, y_3 = y_1 + 1, z_3 = z_1 - 2$$

**Now write $P_2$ & $P_3$ in terms of $x_1, y_1, \& z_1$. Could you see how this same result could be deduced just from looking at the equations of $S_1, S_2, and\ S_3$?**

**Step 3. Since we now have three points on the plane all in terms of $x_1, y_1, \& z_1$, create the two vectors in the plane $P_1P_2\ and\ P_1P_3$.**

**Step 4. Determine the cross-product of the vectors found in Step 3. This vector is a normal vector to the plane. It is also collinear with $\nabla S_1(x_1, y_1, z_1)$.**

**Use that fact as well as $S_1$ to show that $x_1 = -\frac{1}{2}, y_1 = -1,\ and\ z_1 = \frac{5}{4}$**

**Step 5. Using the information from step 4 show that the equation of the tangent plane which is tangent to all three paraboloids is given by:**

$$4x + 8y + 4z + 5 = 0$$