# Mat 241 Homework Set 6key – Due ______________

*Professor David Schultz*

***Directions:*** **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

**Mechanics:**

**#1. If $x^y = y^x$ find $\frac{dy}{dx}$ at $(2,4)$.**

$$x^y = y^x \to \ln\left(x^y\right) = \ln\left(y^x\right) \Leftrightarrow y\ln x = x\ln y$$

$$Let: F(x,y) = y\ln x - x\ln y$$

*then,*

$$\frac{dy}{dx} = -\frac{F_x}{F_y} \to \frac{dy}{dx} = -\frac{\frac{y}{x} - \ln y}{\ln x - \frac{x}{y}} = -\frac{y^2 - xy\ln y}{xy\ln x - x^2} = \frac{xy\ln y - y^2}{xy\ln x - x^2}$$

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \frac{8\ln 4 - 16}{8\ln 2 - 4} = \frac{4\ln 2 - 4}{2\ln 2 - 1}$$

**#2. Given $z^2x^3 + \sin^3\left(yx^2 + yz^3\right) = 2$, find**

$$z^2x^3 + \sin^3\left(yx^2 + yz^3\right) = 2$$

$$Define: F(x,y,z) = z^2x^3 + \sin^3\left(yx^2 + yz^3\right) - 2$$

$$\left(\frac{\partial z}{\partial x}\right)_y = -\frac{F_x}{F_z} = -\frac{3x^2z^2 + 3\sin^2\left(yx^2 + yz^3\right)\cos\left(yx^2 + yz^3\right)2xy}{2zx^3 + 3\sin^2\left(yx^2 + yz^3\right)\cos\left(yx^2 + yz^3\right)3z^2y}$$

$$\left(\frac{\partial z}{\partial y}\right)_x = -\frac{F_y}{F_z} = -\frac{3\sin^2\left(yx^2 + yz^3\right)\cos\left(yx^2 + yz^3\right)\left(x^2 + z^3\right)}{2zx^3 + 3\sin^2\left(yx^2 + yz^3\right)\cos\left(yx^2 + yz^3\right)3z^2y}$$

**#3**. Suppose $z = f(x,y), where\ x = r\cos\theta$ **and** $y = r\sin\theta$
**Show that:**

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial \theta^2} + \frac{1}{r}\frac{\partial z}{\partial r}$$

**Come by my office!**

***Note: This problem is a bit tougher but was assigned when I took the class in 1981.***

**#4. Find the equation of the plane tangent to the surface $z = e^{x^2 - y^2}$ at the point ( 1, -1, 1). Then use Maple to graph both the plane and the surface to verify your result.**

$$z = e^{x^2 - y^2}$$

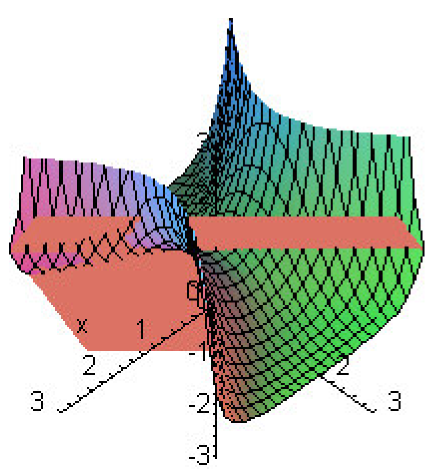
$$Plane_{eq} : z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$z - z_0 = 2x_0 e^{x_0^2 - y_0^2}(x - x_0) - 2y_0 e^{x_0^2 - y_0^2}(y - y_0)$$

$$\Rightarrow z - 1 = 2e^{1-1}(x - 1) + 2e^{1-1}(y + 1)$$

$$= z - 1 = 2x - 2 + 2y + 2$$

$$\therefore 2x + 2y - z + 1 = 0$$



**#5. Compute the partial derivatives problems: 15, 21, 22, and 35 found on page 920.**

$\#15. f(x,y) = xe^{3y}; f_x = e^{3y}; f_y = 3xe^{3y}$

$\#21. f(r,s) = r\ln(r^2+s^2); f_r = \frac{2r^2}{r^2+s^2} + \ln(r^2+s^2); f_y = \frac{2rs}{r^2+s^2}$

$\#22. f(x,t) = \arctan(x\sqrt{t}); f_x = \frac{\sqrt{t}}{1+x^2t}; f_t = \frac{x}{2\sqrt{t}(1+x^2t)}$

$\#35. f(x,y) = \sqrt{x^2+y^2}\, f_x = \frac{x}{\sqrt{x^2+y^2}}; f_y = \frac{y}{\sqrt{x^2+y^2}};$

$f_x(3,4) = \frac{3}{5}, f_y(3,4) = \frac{4}{5}$

**#6. Let $f(x,y) = y\ln x$. Find the gradient of *f*, evaluate the gradient at the point (1, - 3), and then find the rate of change in the direction $\vec{u} = \left\langle -\frac{4}{5}, \frac{3}{5} \right\rangle$.**

$$f(x,y) = y\ln x$$

$$\nabla f(x,y) = \left\langle \frac{y}{x}, \ln x \right\rangle; \nabla f(1,-3) = \langle -3,0 \rangle$$

$$D_{\vec{u}} f(1,-3) = \langle -3,0 \rangle \cdot \left\langle \frac{-4}{5}, \frac{3}{5} \right\rangle = \frac{12}{5}$$

**#7. Find the directional derivative of $f(x,y,z) = x^2+y^2+z^2$ at $P(2,1,3)$ in the direction of the origin.**

$f(x,y,z) = x^2+y^2+z^2$ **at** $P(2,1,3) \to O(0,0,0) \Rightarrow \overrightarrow{PO} = \langle -2,-1,-3 \rangle$

$\nabla f(x,y,z) = \langle 2x,2y,2z \rangle; \nabla f(2,1,3) = \langle 4,2,6 \rangle;$

$$\vec{u}_{\overrightarrow{PO}} = \left\langle \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle$$

$$D_{\vec{u}} f(2,1,3) = \langle 4,2,6 \rangle \cdot \left\langle \frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}} \right\rangle = \frac{-8}{\sqrt{14}} + \frac{-2}{\sqrt{14}} + \frac{-18}{\sqrt{14}} = \frac{-28}{\sqrt{14}}$$

$$= -2\sqrt{14}$$

## Concept Development:

**#8. The temperature (in degrees Celsius at a point (x, y, z) in a metal solid is given by:**

$$T(x,y,z)=\frac{xyz}{1+x^2+y^2+z^2}$$

A. **Find the rate of change of temperature at (1, 1, 1) in the direction of the origin.**

$$T(x,y,z)=\frac{xyz}{1+x^2+y^2+z^2}; Let\ u=1+x^2+y^2+z^2$$

$$\nabla T(x,y,z)=\left\langle \frac{uyz-xyzu_x}{u^2},\frac{uxz-xyzu_y}{u^2},\frac{uxy-xyzu_z}{u^2}\right\rangle$$

$$=\left\langle \frac{uyz-2x^2yz}{u^2},\frac{uxz-2xy^2z}{u^2},\frac{uxy-2xyz^2}{u^2}\right\rangle$$

$$\nabla T(1,1,1)=\left\langle \frac{4-2}{4^2},\frac{4-2}{4^2},\frac{4-2}{4^2}\right\rangle=\left\langle \frac{1}{8},\frac{1}{8},\frac{1}{8}\right\rangle$$

$$\vec{u}_{\overline{PO}}=\left\langle \frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right\rangle$$

$$D_{\vec{u}}T(1,1,1)=\left\langle \frac{1}{8},\frac{1}{8},\frac{1}{8}\right\rangle\cdot\left\langle \frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}},\frac{-1}{\sqrt{3}}\right\rangle=\frac{-3}{8\sqrt{3}}=\frac{-\sqrt{3}}{8}$$

B. **Find the direction in which the temperature rises most rapidly at (1, 1, 1). Express your answer as a unit vector.**

***The gradient gives the direction of greatest increase at any point, P, in the function's domain.***

$$\nabla T(1,1,1)=\left\langle \frac{1}{8},\frac{1}{8},\frac{1}{8}\right\rangle\ and\ \vec{u}_{\nabla T(1,1,1)}=\frac{\left\langle \frac{1}{8},\frac{1}{8},\frac{1}{8}\right\rangle}{\sqrt{\frac{1}{64}+\frac{1}{64}+\frac{1}{64}}}=\frac{8}{\sqrt{3}}\left\langle \frac{1}{8},\frac{1}{8},\frac{1}{8}\right\rangle$$

$$=\left\langle \frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle\ or\ \left\langle \frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3},\frac{\sqrt{3}}{3}\right\rangle$$

C. **Find the rate at which the temperature rises moving from (1, 1, 1) in the direction found in part B.**

***This will simply be the norm of the gradient.***

$$\nabla \boldsymbol{T}(1,1,1) = \left\langle \frac{1}{8}, \frac{1}{8}, \frac{1}{8} \right\rangle \Rightarrow \left\| \nabla \boldsymbol{T}(1,1,1) \right\| = \frac{\sqrt{3}}{8}$$

**#9. A certain bug happened to land on a flat plate whose temperature at a point (x, y) is give by $\boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}) = 5 + 2\boldsymbol{x}^2 + \boldsymbol{y}^2$.**

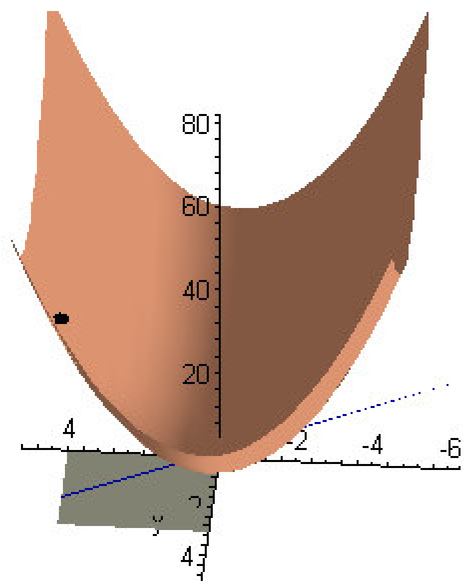
A. **Determine the direction the pest should take, starting at (4, 2), in order to cool off as rapidly as possible.**

$$\boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}) = 5 + 2\boldsymbol{x}^2 + \boldsymbol{y}^2$$

$$\nabla \boldsymbol{T}(\boldsymbol{x},\boldsymbol{y}) = \langle 4\boldsymbol{x}, 2\boldsymbol{y} \rangle \to \nabla \boldsymbol{T}(4,2) = \langle 16, 4 \rangle = 16\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}$$

$$\therefore \textbf{maximum decrease direction: } -\nabla \boldsymbol{T}(4,2) = -16\vec{\boldsymbol{i}} - 4\vec{\boldsymbol{j}}$$

B. **Plot the temperature function surface *T* using Maple.**




**#10. let $f_x, f_y, f_{xx}, f_{yy}$ be continuous and $\vec{u}$ and $\vec{v}$ are constant unit vectors given by $\vec{u} = \langle u_1, u_2 \rangle$ & $\vec{v} = \langle v_1, v_2 \rangle$. Show that $D_{\vec{u}}(D_{\vec{v}} f) = D_{\vec{v}}(D_{\vec{u}} f)$. *Where* $D_{\vec{u}} f = \nabla f \cdot \vec{u}$ and $D_{\vec{v}} f = \nabla f \cdot \vec{v}$.**

$D_{\vec{u}}(D_{\vec{v}} f) = D_{\vec{u}}(\nabla f \cdot \vec{v}) = D_{\vec{u}}(\langle f_x, f_y \rangle \cdot \langle v_1, v_2 \rangle) = D_{\vec{u}}(f_x v_1 + f_y v_2)$

$= \langle f_x(f_x v_1 + f_y v_2), f_y(f_x v_1 + f_y v_2) \rangle \cdot \langle u_1, u_2 \rangle$

$= \langle f_{xx} v_1 + f_{yx} v_2, f_{xy} v_1 + f_{yy} v_2 \rangle \cdot \langle u_1, u_2 \rangle$

$= f_{xx} v_1 u_1 + f_{yx} v_2 u_1 + f_{xy} v_1 u_2 + f_{yy} v_2 u_2$

$= f_{xx} u_1 v_1 + f_{yx} u_1 v_2 + f_{xy} u_2 v_1 + f_{yy} u_2 v_2$ /* *commutative law*

$= \langle f_{xx} u_1 + f_{yx} u_2, f_{xy} u_1 + f_{yy} u_2 \rangle \cdot \langle v_1, v_2 \rangle$ /* *dot product*

$= \langle f_x(f_x u_1 + f_y u_2), f_y(f_x u_1 + f_y u_2) \rangle \cdot \langle v_1, v_2 \rangle$ /* *definition partial derivatives*

$= D_{\vec{v}}(f_x u_1 + f_y u_2)$ /* *definition directional derivative*

$= D_{\vec{v}}(\langle f_x, f_y \rangle \cdot \langle u_1, u_2 \rangle)$ /* *dot product*

$= D_{\vec{v}}(\nabla f \cdot \vec{u})$ /* *definition directional derivative*

$= D_{\vec{v}}(D_{\vec{u}} f)$ /* *given*