

# Mat 241 Homework Set 11key – Due \_\_\_\_\_

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**Directions:** Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.

## **Mechanics:**

#1. On page 1060 of our text do pbs. 12-18 even.

12 – IV since the x- component of each vector is constant, the vectors are independent of x.

14 – III the vector directions in the individual quadrants match up.

16 – I horizontal vectors are constant and ones above and below xy – plane point in opposite directions.

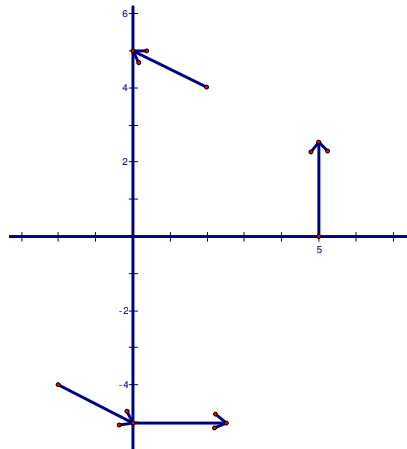
18 – II Each vector has the same magnitude and direction as the position vector, and hence, all point away from the origin.

#2. Find the gradient vector field for  $f(x, y, z) = x \cos\left(\frac{y}{z}\right)$ .

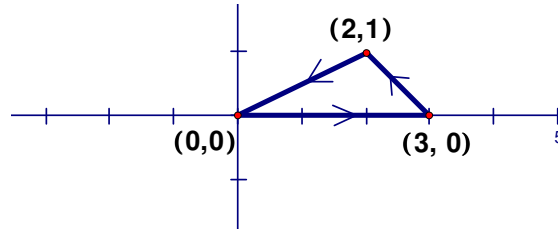
$$\bar{\nabla} \cdot f = \left\langle \cos\left(\frac{y}{z}\right), -\frac{x}{z} \sin\left(\frac{y}{z}\right), \frac{xy}{z^2} \sin\left(\frac{y}{z}\right) \right\rangle$$

#3. Given the vector field  $\bar{F}(x, y) = \left\langle -\frac{y}{2}, \frac{x}{2} \right\rangle$ , do the following:

A. Sketch the field vectors corresponding to the points (2, 4), (-2, -4), (5, 0) and (0, -5).



B. Evaluate the line integral  $\oint_C \vec{F} \cdot d\vec{r}$  over the two shown paths.



$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r}$$

$$\oint_{C_1} \vec{F} \cdot d\vec{r} : x(t) = 3t, y(t) = 0; \vec{r}(t) = \langle 3t, 0 \rangle; \vec{r}'(t) = \langle 3, 0 \rangle;$$

$$\vec{F}(\vec{r}(t)) = \left\langle 0, \frac{3t}{2} \right\rangle$$

$$\oint_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \left\langle 0, \frac{3t}{2} \right\rangle \cdot \langle 3, 0 \rangle dt = \int_0^1 0 dt = 0$$

$$\oint_{C_2} \vec{F} \cdot d\vec{r} : x(t) = 3-t, y(t) = t; \vec{r}(t) = \langle 3-t, t \rangle; \vec{r}'(t) = \langle -1, 1 \rangle;$$

$$\vec{F}(\vec{r}(t)) = \left\langle \frac{-t}{2}, \frac{3-t}{2} \right\rangle$$

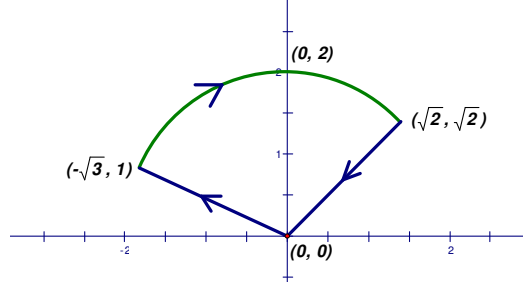
$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \int_0^1 \left\langle \frac{-t}{2}, \frac{3-t}{2} \right\rangle \cdot \langle -1, 1 \rangle dt = \int_0^1 \frac{3}{2} dt = \frac{3}{2}$$

$$\oint_{C_3} \vec{F} \cdot d\vec{r} : x(t) = 2-t, y(t) = 1-\frac{t}{2}; \vec{r}(t) = \left\langle 2-t, 1-\frac{t}{2} \right\rangle; \vec{r}'(t) = \left\langle -1, -\frac{1}{2} \right\rangle;$$

$$\vec{F}(\vec{r}(t)) = \left\langle \frac{-1}{2} + \frac{t}{4}, 1-\frac{t}{2} \right\rangle$$

$$\oint_{C_3} \vec{F} \cdot d\vec{r} = \int_0^2 \left\langle \frac{-1}{2} + \frac{t}{4}, 1-\frac{t}{2} \right\rangle \cdot \left\langle -1, -\frac{1}{2} \right\rangle dt = \int_0^2 0 dt = 0$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} = \frac{3}{2}$$



$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r}; \oint_{C_1} \vec{F} \cdot d\vec{r} : x(t) = -\sqrt{3}t, y(t) = t;$$

$$\vec{r}(t) = \langle -\sqrt{3}t, t \rangle; \vec{r}'(t) = \langle -\sqrt{3}, 1 \rangle;$$

$$\vec{F}(\vec{r}(t)) = \left\langle \frac{-t}{2}, \frac{-\sqrt{3}t}{2} \right\rangle; \oint_{C_1} \vec{F} \cdot d\vec{r} = \int_0^1 \left\langle \frac{-t}{2}, \frac{-\sqrt{3}t}{2} \right\rangle \cdot \langle -\sqrt{3}, 1 \rangle dt = \int_0^1 0 dt = 0$$

$$\oint_{C_2} \vec{F} \cdot d\vec{r} : x(t) = 2\cos(\pi - t), y(t) = 2\sin(\pi - t); \vec{r}(t) = \langle 2\cos(\pi - t), 2\sin(\pi - t) \rangle;$$

$$\vec{r}'(t) = \langle 2\sin t, -2\cos t \rangle; \frac{\pi}{4} \leq t \leq \frac{5\pi}{6}; \vec{F}(\vec{r}(t)) = \langle -\sin(\pi - t), \cos(\pi - t) \rangle$$

$$\oint_{C_2} \vec{F} \cdot d\vec{r} = \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} \langle -\sin(\pi - t), \cos(\pi - t) \rangle \cdot \langle 2\sin(\pi - t), -2\cos(\pi - t) \rangle dt$$

$$= -2 \int_{\frac{\pi}{4}}^{\frac{5\pi}{6}} 1 dt = -2t \Big|_{\frac{\pi}{4}}^{\frac{5\pi}{6}} = -2 \left( \frac{5\pi}{6} - \frac{\pi}{4} \right) = -2 \frac{14\pi}{24} = -\frac{7\pi}{6}$$

$$\oint_{C_3} \vec{F} \cdot d\vec{r} : x(t) = \sqrt{2}(1-t), y(t) = \sqrt{2}(1-t); \vec{r}(t) = \langle \sqrt{2}(1-t), \sqrt{2}(1-t) \rangle;$$

$$\vec{r}'(t) = \langle -\sqrt{2}, -\sqrt{2} \rangle; \vec{F}(\vec{r}(t)) = \left\langle -\frac{\sqrt{2}(1-t)}{2}, \frac{\sqrt{2}(1-t)}{2} \right\rangle$$

$$\oint_{C_3} \vec{F} \cdot d\vec{r} = \int_0^1 \left\langle -\frac{\sqrt{2}(1-t)}{2}, \frac{\sqrt{2}(1-t)}{2} \right\rangle \cdot \langle -\sqrt{2}, -\sqrt{2} \rangle dt = \int_0^1 0 dt = 0$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{C_2} \vec{F} \cdot d\vec{r} + \oint_{C_3} \vec{F} \cdot d\vec{r} = -\frac{7\pi}{6}$$

#3. Using the force  $\vec{F} = \langle 2x, -2yz, -y^2 \rangle$  do the following:

- A. Determine a scalar function  $f(x, y, z)$  such that  $\vec{F} = \nabla f$ . The function  $f$  is called a *potential function*.

$$f(x, y, z) = x^2 - y^2z + k$$

- B. Verify that  $\vec{F}$  is conservative by showing that  $\oint_C \vec{F} \cdot d\vec{r} = 0$  where  $C$  is

composed of  $C_1$  which is a portion of the helix

$\vec{r}(t) = \langle \cos t, \sin t, t \rangle$  from  $(1, 0, 0)$  to  $\left(0, -1, \frac{3\pi}{2}\right)$  and  $C_2$  is the line

segment from  $\left(0, -1, \frac{3\pi}{2}\right)$  to  $(1, 0, 0)$ .

$$\vec{r}_{C_1}(t) = \langle \cos t, \sin t, t \rangle; \vec{r}'_{C_1}(t) = \langle -\sin t, \cos t, 1 \rangle;$$

$$\vec{F}(t) = \langle 2\cos t, -2t \sin t, -\sin^2 t \rangle$$

$$\vec{F}(t) \cdot \vec{r}'_{C_1}(t) = \langle 2\cos t, -2t \sin t, -\sin^2 t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle$$

$$= -\sin(2t) - t \sin(2t) - \sin^2 t$$

$$\int_{C_1} \vec{F}(t) \cdot \vec{r}'_{C_1}(t) dt = \int_0^{\frac{3\pi}{2}} (-\sin(2t) - t \sin(2t) - \sin^2 t) dt$$

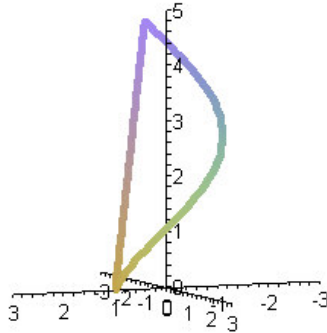
$$= -\int_0^{\frac{3\pi}{2}} \sin(2t) dt - \int_0^{\frac{3\pi}{2}} t \sin(2t) dt - \int_0^{\frac{3\pi}{2}} \sin^2(t) dt$$

$$= \frac{1}{2} \cos u \Big|_0^{3\pi} - \frac{1}{4} \left[ -u \cos u \Big|_0^{3\pi} + \int_0^{3\pi} \cos u du \right] - \int_0^{\frac{3\pi}{2}} \sin^2 t dt$$

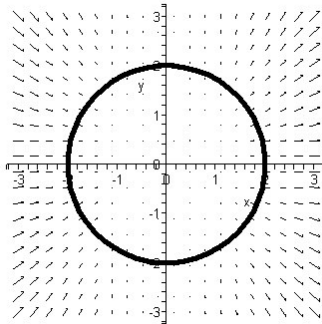
$$= -1 - \frac{1}{4} \left[ 3\pi + \cancel{\sin u} \Big|_0^{3\pi} \right] - \left[ \frac{t}{2} - \frac{1}{4} \sin(2t) \Big|_0^{\frac{3\pi}{2}} \right]$$

$$= -1 - \frac{3\pi}{4} - \frac{3\pi}{4} = -1 - \frac{3\pi}{2}$$

$$\begin{aligned} \vec{r}(t) &= \left\langle t, t-1, -\frac{3\pi}{2}(t-1) \right\rangle; \vec{r}'(t) = \left\langle 1, 1, -\frac{3\pi}{2} \right\rangle; \\ \vec{F}(\vec{r}(t)) &= \left\langle 2t, 3\pi(t-1)^2, -(t-1)^2 \right\rangle \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= \left\langle 2t, 3\pi(t-1)^2, -(t-1)^2 \right\rangle \cdot \left\langle 1, 1, -\frac{3\pi}{2} \right\rangle \\ &= 2t + 3\pi(t-1)^2 + \frac{3\pi}{2}(t-1)^2 \\ \int_{c_2} \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt &= \int_0^1 \left( 2t + 3\pi(t-1)^2 + \frac{3\pi}{2}(t-1)^2 \right) dt \\ &= t^2 + \pi(t-1)^3 + \frac{3\pi}{6}(t-1)^3 \Big|_0^1 = 1 + \pi + \frac{\pi}{2} = 1 + \frac{3\pi}{2} \\ \oint_c \vec{F} \cdot d\vec{r} &= \int_{c_1} \vec{F} \cdot d\vec{r} + \int_{c_2} \vec{F} \cdot d\vec{r} = -1 - \frac{3\pi}{2} + 1 + \frac{3\pi}{2} = 0 \end{aligned}$$



#4. Find the work done by the force field  $\vec{F}(x, y) = x^2\vec{i} + xy\vec{j}$  on a particle that moves once around a circle centered at the origin of radius 4 oriented in the counterclockwise direction. How does your result play itself out in the picture?

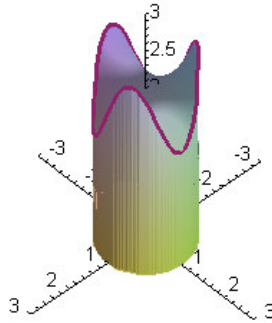


$$\begin{aligned}\vec{r}(t) &= \langle 2\cos t, 2\sin t \rangle; \vec{r}'(t) = \langle -2\sin t, 2\cos t \rangle; \vec{F}(\vec{r}(t)) = \langle 4\cos^2 t, 4\cos t \sin t \rangle \\ \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) &= \langle 4\cos^2 t, 4\cos t \sin t \rangle \cdot \langle -2\sin t, 2\cos t \rangle = -8\cos^2 t \sin t + 8\cos^2 t \sin t \\ &= 0\end{aligned}$$

$$\oint_C \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t) dt = \int_0^{2\pi} 0 dt = 0$$

∴ We can see that the symmetry and the perpendicular nature of the force vectors to the curve yield zero work.

#5. A cylindrical solid piece of bar-stock of radius 1 inch is cut so that the height along the top edge is given by the curve:  $z = 2 + 0.5\sin(3t)$   $0 \leq t \leq 2\pi$ . (Curve is shown running along the edge of the solid).



Do the following.

1. Determine the lateral surface area by means of a geometric argument.

The sinusoidal nature of the defining curve has a midline of  $z = 2$ . So, we may expect the surface area to be equivalent to the surface area of a cylinder of radius 1 inch and height of 2 inches. Thus,  $SA = 2\pi rh = 4\pi \text{ in}^2$

2. Determine the lateral surface area using a line integral.

$$x(t) = \cos t, x'(t) = -\sin t; y(t) = \sin t; y'(t) = \cos t$$

$$\begin{aligned}\int_0^{2\pi} (2 + 0.5\sin(3t))\sqrt{(-\sin t)^2 + (\cos t)^2} dt &= \int_0^{2\pi} (2 + 0.5\sin(3t)) dt = 2t - \frac{\cos(3t)}{6} \Big|_0^{2\pi} \\ &= \left(4\pi - \frac{1}{6}\right) - \left(0 - \frac{1}{6}\right) = 4\pi \text{ in}^2\end{aligned}$$

3. Suppose you wanted to plate the lateral surface area with nickel plating 1 millimeter thick. According to Prime Plating Co. of Sun Valley, Ca. it costs about 6 cents to plate 1 square inch at that thickness. What would the approximate cost be?

$$Cost = 2\pi \times 0.06 \approx \text{\$0.75}$$

4. Show that the curve  $z = 2 + 0.5\sin(3t)$  which traces out the edge of the solid's top lies on the surface  $z = 2 + x^2y + 0.5y(x^2 - y^2)$  by using trigonometric identities learned in MAT 182 on  $2 + 0.5\sin(3t)$ . You should enjoy this!

$$\begin{aligned} & 2 + 0.5\sin(3t) \\ &= 2 + 0.5\sin(2t + t) \\ &= 2 + 0.5[\sin(2t)\cos t + \sin t \cos(2t)] \\ &= 2 + 0.5[2\sin t \cos^2 t + \sin t(\cos^2 t - \sin^2 t)] \\ &= 2 + 0.5[2yx^2 + y(x^2 - y^2)] \\ &= 2 + yx^2 + y(x^2 - y^2) \end{aligned}$$

5. Just for fun...Returning to question 3, suppose you wanted to plate the entire piece's surface. Top, bottom, and lateral side. What would that cost?

$$\begin{aligned} Cost &= Cost_{bottom} + Cost_{lateral} + Cost_{top} \\ &= \$0.06/in^2 \left( \pi in^2 + 4\pi in^2 + \int_{-1-\sqrt{1-x^2}}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^4 + 4x^2y^2 + 9y^4 + 1} dydx \right) \end{aligned}$$

*Maple struggles!*

More to come.....

#6. This last problem is from my homework set in 1982. If  $\gamma$  is a smooth, but not necessarily simple, closed curve which does not pass through the origin, then the winding number of  $\gamma$  is defined as

$$\frac{1}{2\pi} \int_{\gamma} \bar{F} \cdot d\bar{r} \quad \text{where } \bar{F}(x, y) = \left( \frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2} \right).$$

It can be shown in advanced calculus texts that the winding number is always an integer which gives the number of times  $\gamma$  winds around the origin. If it is positive, then  $\gamma$  encircles the origin in a counterclockwise sense. If it is negative, then  $\gamma$  encircles the origin in a clockwise sense. Compute the winding numbers for the following two curves.

$$\gamma_1 : \bar{r}(t) = \langle 2\cos t, 2\sin t \rangle \quad 0 \leq t \leq 6\pi$$

and

$$\gamma_2 : \bar{r}(t) = \langle \cos t, -\sin t \rangle \quad 0 \leq t \leq 2\pi$$

$$\bar{r}(t) = \langle 2\cos t, 2\sin t \rangle; \bar{r}'(t) = \langle -2\sin t, 2\cos t \rangle; \bar{F}(\bar{r}(t)) = \left\langle \frac{-\sin t}{2}, \frac{\cos t}{2} \right\rangle$$

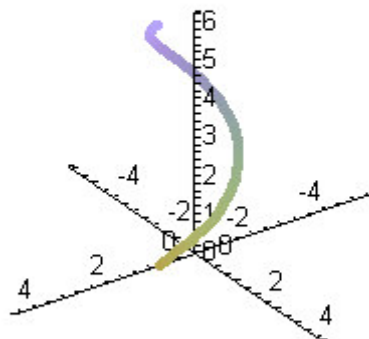
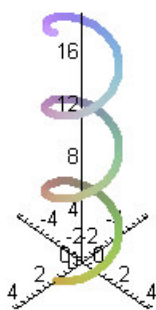
$$\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) = \left\langle \frac{-\sin t}{2}, \frac{\cos t}{2} \right\rangle \cdot \langle -2\sin t, 2\cos t \rangle = \sin^2 t + \cos^2 t = 1$$

$$\frac{1}{2\pi} \int_C \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) dt = \frac{1}{2\pi} \int_0^{6\pi} dt = 3$$

$$\bar{r}(t) = \langle \cos t, -\sin t \rangle; \bar{r}'(t) = \langle -\sin t, -\cos t \rangle; \bar{F}(\bar{r}(t)) = \langle \sin t, \cos t \rangle$$

$$\bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) = \langle \sin t, \cos t \rangle \cdot \langle -\sin t, -\cos t \rangle = -\sin^2 t - \cos^2 t = -1$$

$$\frac{1}{2\pi} \int_C \bar{F}(\bar{r}(t)) \cdot \bar{r}'(t) dt = -\frac{1}{2\pi} \int_0^{2\pi} dt = -1$$





**Electroless Nickel:** Electroless nickel (EN) plating is a process that utilizes an autocatalytic chemical reaction instead of electricity to deposit nickel ions onto a part. This type of plating process gives uniform thickness all over the part's shape and size without the variances that occur in electrolytic plating. This offers a distinct advantage for parts that require very exact tolerances and also offers superior corrosion resistance to electroplated nickel or hard chrome.