Mat 241 Homework Set 1 Key – Due ______________

Professor David Schultz

Directions: **Show *all algebraic* steps neatly and concisely using *proper mathematical symbolism*. When graphs and technology are to be implemented, do so appropriately.**

Mechanics:

#1. Show that the following equation represents a sphere and determine its center and radius.

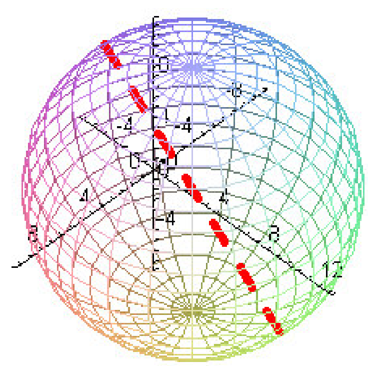
$$3x^2 - 6x + 3y^2 + 3z^2 + 12z - 6 = 0$$
$$3\left(x^2 - 2x\right) + 3y^2 + 3\left(z^2 + 4z\right) = 6$$
$$\left(x^2 - 2x\right) + y^2 + \left(z^2 + 4z\right) = 2$$
$$\left(x-1\right)^2 - 1 + y^2 + \left(z+2\right)^2 - 4 = 2$$
$$\left(x-1\right)^2 + y^2 + \left(z+2\right)^2 = 7$$
$$\therefore center: \left(1,0,-2\right);\ radius: \sqrt{7}$$

#2. Find the equation of a sphere if one of its diameters has endpoints given by $\left(0,-4,7\right)$ **&** $\left(2,12,-3\right)$**.**

$$Sphere_{center}: \left(\frac{2+0}{2}, \frac{12+(-4)}{2}, \frac{-3+7}{2}\right) = \left(1,4,2\right)$$

$$Radius: r = \sqrt{\left(2-1\right)^2 + \left(12-4\right)^2 + \left(-3-2\right)^2} = \sqrt{1+64+25} = \sqrt{90}$$

$$Sphere_{equation:} \left(x-1\right)^2 + \left(y-4\right)^2 + \left(z-2\right)^2 = 90$$



#3. Given the vectors $\vec{a} = \langle a_1, a_2 \rangle$ and $\vec{b} = \langle b_1, b_2 \rangle$ and the scalar, k, use vector addition and scalar multiplication to show that: $k\left(\vec{a} + \vec{b}\right) = k\vec{a} + k\vec{b}$ (a kind of distributive law). Make sure you show and justify each step.

$$\begin{aligned}
k\left(\vec{a} + \vec{b}\right) &= k\left(\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle\right) \text{ /* } \textit{given} \\
&= k\left(\langle a_1 + b_1, a_2 + b_2 \rangle\right) \text{ /* } \textit{vector addition} \\
&= \langle k(a_1 + b_1), k(a_2 + b_2) \rangle \text{ /* } \textit{scalar multiplication} \\
&= \langle ka_1 + kb_1, ka_2 + kb_2 \rangle \text{ /* } \textit{distributive property} \\
&= \langle ka_1, ka_2 \rangle + \langle kb_1, kb_2 \rangle \text{ /* } \textit{vector addition} \\
&= k\langle a_1, a_2 \rangle + k\langle b_1, b_2 \rangle \text{ /* } \textit{scalar multiplication} \\
&= k\vec{a} + k\vec{b} \text{ /* } \textit{given}
\end{aligned}$$

#4. Given the points $P_1(1,1,7)$ & $P_2(4,7,5)$ find the vectors $\overrightarrow{P_1P_2}$ & $\overrightarrow{P_2P_1}$. Compare the two results.

$$\overrightarrow{P_1P_2} = \langle 4-1, 7-1, 5-7 \rangle = \langle 3, 6, -2 \rangle$$

$$\overrightarrow{P_2P_1} = \langle 1-4, 1-7, 7-5 \rangle = \langle -3, -6, 2 \rangle = -1\overrightarrow{P_1P_2}$$

#5. Given the vectors $\vec{a} = \langle 1, 2, -2 \rangle, \vec{b} = \langle 2, -3, 1 \rangle$ *and* $\vec{c} = \langle -2, 3, 6 \rangle$ do the following.

A. Determine: $\vec{d} = 2\vec{a} - \vec{b}$.

$$\vec{d} = 2\vec{a} - \vec{b} = 2\langle 1, 2, -2 \rangle - \langle 2, -3, 1 \rangle = \langle 2, 4, -4 \rangle - \langle 2, -3, 1 \rangle = \langle 0, 7, -5 \rangle$$

B. Determine the unit vector in the direction of $\vec{c}$ (i.e. find $\vec{u}_c$).

$$\vec{c} = \langle -2, 3, 6 \rangle; |\vec{c}| = \sqrt{(-2)^2 + 3^2 + 6^2} = 7$$

$$\vec{u}_{\vec{c}} = \frac{1}{7}\langle -2, 3, 6 \rangle = \left\langle \frac{-2}{7}, \frac{3}{7}, \frac{6}{7} \right\rangle$$

C. Determine a vector that points in the opposite direction of $\vec{c}$ and has a magnitude of 10.

We seek $\vec{c}^* = -10\vec{u}_{\vec{c}}$. So, $-10\vec{u}_{\vec{c}} = \frac{-10}{7}\langle -2, 3, 6 \rangle = \left\langle \frac{20}{7}, \frac{-30}{7}, \frac{-60}{7} \right\rangle$

D. **Determine a vector that points in the same direction as $\vec{c}$ but is one-half as long.**

$$\textbf{We seek } \frac{1}{2}\vec{c} = \frac{1}{2}\langle -2,3,6 \rangle = \left\langle -1,\frac{3}{2},3 \right\rangle$$

<u>Concept Development:</u>

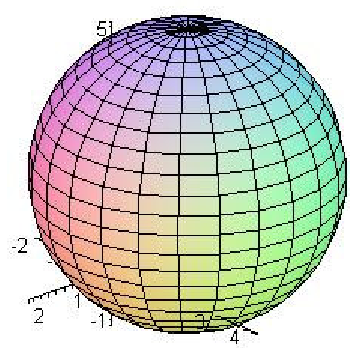
#6. If a bug walks on the sphere $x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$, how close and how far can he get from the origin?

$$x^2 + y^2 + z^2 + 2x - 2y - 4z - 3 = 0$$
$$(x+1)^2 - 1 + (y-1)^2 - 1 + (z-2)^2 - 4 = 3$$
$$(x+1)^2 + (y-1)^2 + (z-2)^2 = 9$$
$$\therefore \textbf{center}: (-1,1,2); \textbf{ radius}: 3$$

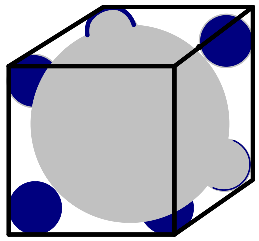
The distance from the center of the circle to the origin is:

$$d = \sqrt{(-1-0)^2 + (1-0)^2 + (2-0)^2} = \sqrt{6}$$

Since $\sqrt{6} < 3$, we know that the origin lies inside the sphere. Hence the closest the bug can get is $3 - \sqrt{6}$ and the furthest is $3 + \sqrt{6}$.



#7. As shown in the accompanying figure, a bowing ball of radius, R, is placed inside a box just large enough to contain it. In order to protect it during shipping 8 little Styrofoam Balls are placed in each corner to absorb shock. What is the radius, r, of the largest Styrofoam ball to be used?



The distance from the origin to the center of the bowling ball is $d = \sqrt{R^2 + \left(\sqrt{2}R\right)^2} = \sqrt{3}R$. This same distance can also be written as: $d = \sqrt{3}r + r + R$. Equating and solving for little r yields:

$$\sqrt{3}r + r + R = \sqrt{3}R \Leftrightarrow \sqrt{3}r + r = \sqrt{3}R - R$$

$$r\left(\sqrt{3}+1\right) = R\left(\sqrt{3}-1\right)$$

$$r = \left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)R = \left(2-\sqrt{3}\right)R$$

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> restart:with(plottools):with(plots):
> c := sphere([1,1,1],1):d:=sphere([2-sqrt(3),2-sqrt(3),2-sqrt(3)],2-
sqrt(3)):b1 := arrow([0,0,0], [1,1,1],.1,.1,.1, color=red):b2 :=
arrow([0,0,0], [1,1,0],.1,.1,.1, color=green):b3 := arrow([1,1,0],
[1,1,1],.1,.1,.1, color=blue):plots[display](c,b1,d,b2,
b3,scaling=constrained,style=wireframe,axes=boxed,orientation=[149,58])
;
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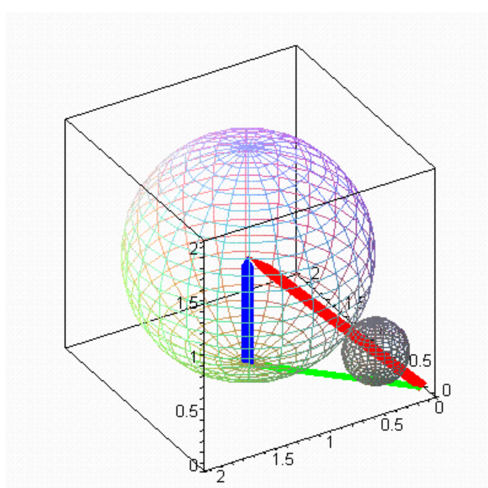


#8. Consider the parabola $y = x^2$. Find two unit vectors that are parallel to the tangent line at the point (2, 4). I want you to sketch accurately the parabola, tangent line, and two unit vectors on the same graph.

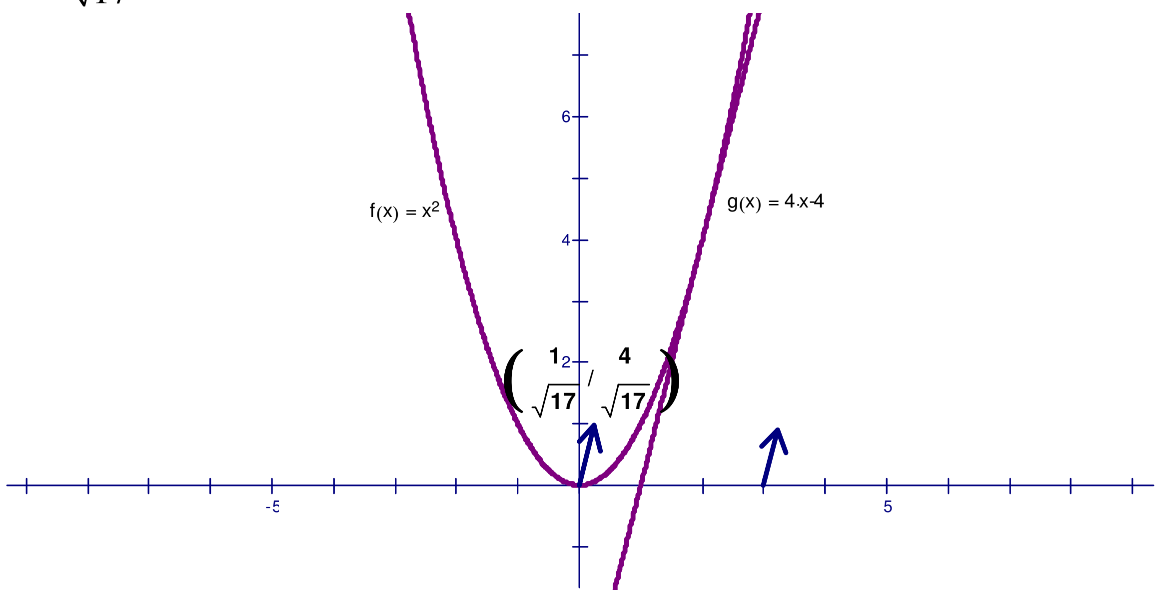
$$y = x^2$$

$$y' = 2x; y'(2) = 4$$

$$y_{\tan} = 4x - 4$$

$$m = \frac{4}{1}$$

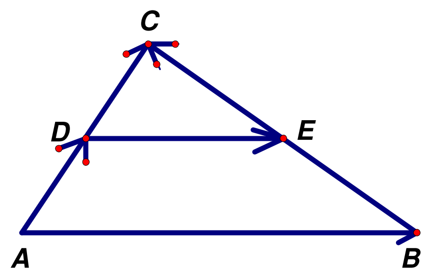
$$\vec{u} = \frac{1}{\sqrt{17}}\langle 1,4 \rangle:$$



Note: Answer is not unique.

#9. Use vectors to prove that the line segment joining the midpoints of two sides of a triangle is parallel to the third side and half its length. Provide a sketch with the vectors labeled appropriately.

Consider the figure shown:



$\overrightarrow{BC} = \overrightarrow{AC} - \overrightarrow{AB}$ /*1

$\overrightarrow{EC} = \overrightarrow{DC} - \overrightarrow{DE}$ **but** $\overrightarrow{EC} = \frac{1}{2}\overrightarrow{BC}$ **and** $\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AC}$

Thus **:**

$\overrightarrow{EC} = \overrightarrow{DC} - \overrightarrow{DE} \Leftrightarrow \frac{1}{2}\overrightarrow{BC} = \frac{1}{2}\overrightarrow{AC} - \overrightarrow{DE} \Leftrightarrow \overrightarrow{BC} = \overrightarrow{AC} - 2\overrightarrow{DE}$ /*2.

From **/*1 &/*2 we have:**

$\overrightarrow{AC} - \overrightarrow{AB} = \overrightarrow{AC} - 2\overrightarrow{DE}$

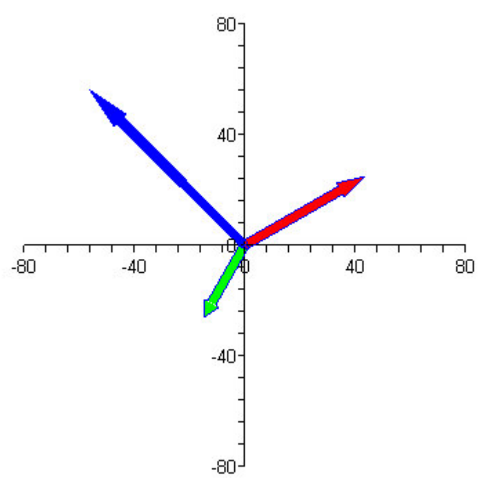
$\overrightarrow{AB} = 2\overrightarrow{DE}$

$\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AB}$

Since one vector is a scalar multiple of the other they must be parallel and the scalar value confirms that the line segment in question is half the third side's length.

#10. Find the resultant of the following three forces: 1. a force of 50 N making an angle of $30°$ with the +x-axis. 2. a force of 80 N making an angle of $135°$ with the + x-axis, and a force of 30 N making an angle of $240°$ with the +x-axis.

A. Draw each vector in standard position on the same coordinate system.

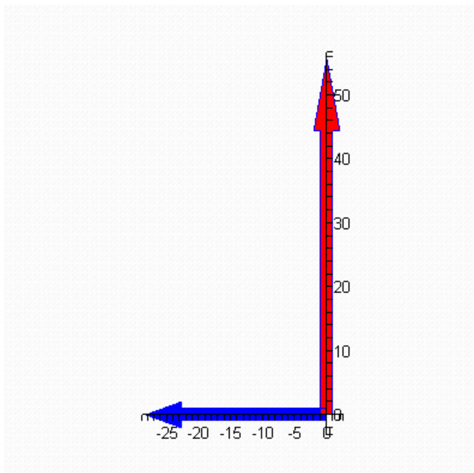


B. Resolve each vector into its components.

$$F_1 = |F_1|\cos 30°\vec{i} + |F_1|\sin 30°\vec{j};$$
$$F_2 = -|F_2|\cos 45°\vec{i} + |F_2|\sin 45°\vec{j};$$
$$F_3 = -|F_3|\cos 60°\vec{i} - |F_3|\sin 60°\vec{j}$$

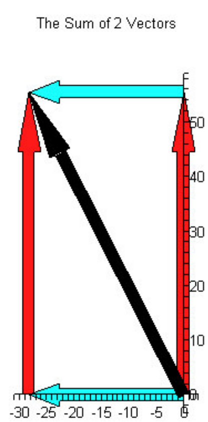
C. Determine the components of the resultant vector and sketch them on a new coordinate system in standard position.

$$\vec{r} = \left(|F_1|\cos 30° - |F_2|\cos 45° - |F_3|\cos 60°\right)\vec{i} + \left(|F_1|\sin 30° + |F_2|\sin 45° - |F_3|\sin 60°\right)\vec{j}$$
$$= \left(50\frac{\sqrt{3}}{2} - 80\frac{\sqrt{2}}{2} - 30\frac{1}{2}\right)\vec{i} + \left(50\frac{1}{2} + 80\frac{\sqrt{2}}{2} - 30\frac{\sqrt{3}}{2}\right)\vec{j}$$
$$= \left(25\sqrt{3} - 40\sqrt{2} - 15\right)\vec{i} + \left(25 + 40\sqrt{2} - 15\sqrt{3}\right)\vec{j}$$
$$\approx -28.27\vec{i} + 55.59\vec{j}$$

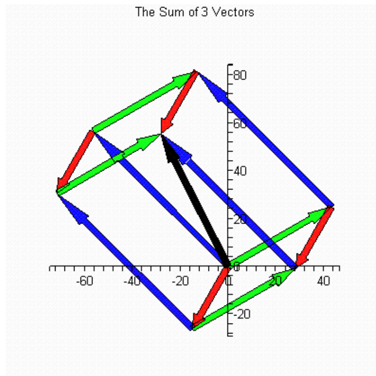


D. Draw the resultant vector, $\vec{r}$, and also determine $|\vec{r}|$ *and* $\theta_{\vec{r}}$.

$$|\vec{r}| = \sqrt{(-28.27)^2 + 55.59^2} \approx 62.37; \quad m\angle\theta = 180^{\circ} + \tan^{-1}\left(\frac{55.59}{-28.27}\right) \approx 116.96^{\circ}$$

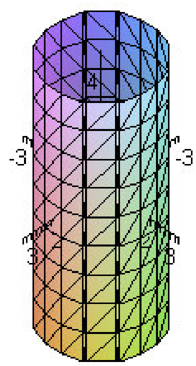


```
> with(Student[LinearAlgebra]):
> infolevel[Student[LinearAlgebra]] := 1:
> VectorSumPlot( <50*cos(Pi/6),50*sin(Pi/6)>, <-
80*cos(Pi/4),80*sin(Pi/4)>,<-30*cos(Pi/3),-30*sin(Pi/3)>);
sum = <-28.27, 55.59>
```

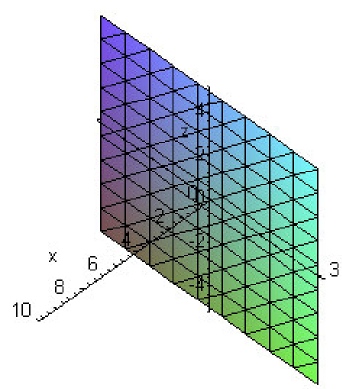


#11. Sketch on grid paper accurately the following 3-D drawings.

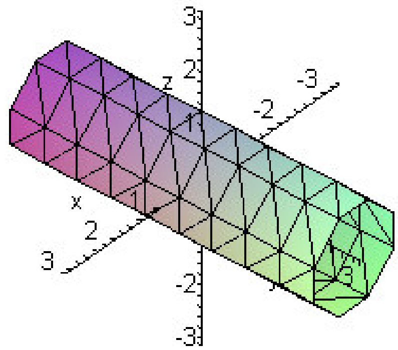
A. A circle in the xy-plane of radius 2.



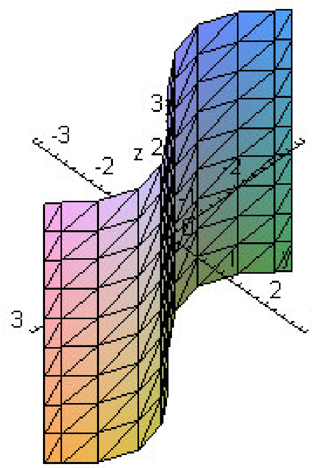
B. The yz-plane



C. The surface $x^2 + z^2 = 1$

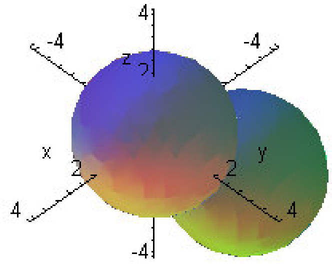


D. The surface y = sinx .



#12.Computer Graphics Project 1:

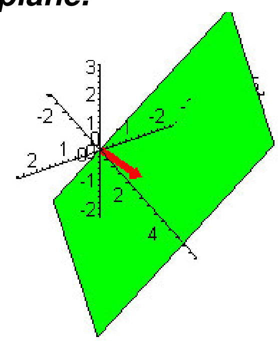
Using a 3-D Graphing Software tool, plot the spheres:
$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0$ **and** $x^2 + y^2 + z^2 = 4$**.**



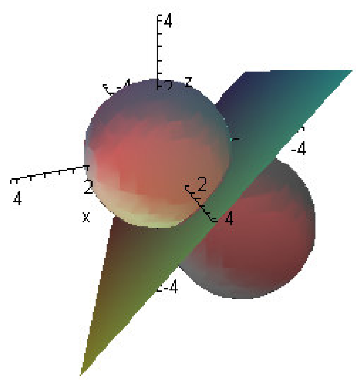
A. **Determine the equation for the intersection of these two spheres.**

$$x^2 + y^2 + z^2 + 4x - 2y + 4z + 5 = 0 \;\&\; x^2 + y^2 + z^2 + 4$$
$$4 + 4x - 2y + 4z + 5 = 0$$
$$4x - 2y + 4z + 9 = 0$$

B. **What shape is it?** ***A plane.***



C. **Add the equation to part A in your graph.**



D. **Bonus Challenge: What is the volume contained in their intersection?**

Answer in office only

#13. Practice with integration. Compute the following integrals. Sketch the region for the definite integrals.

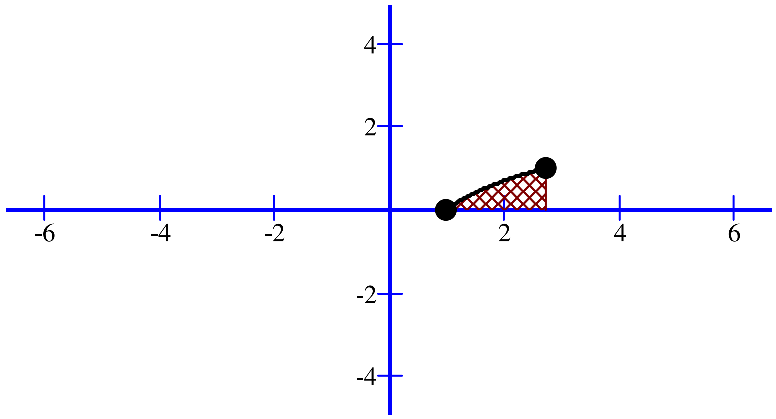
A.

$$\int_1^e \ln(x)dx \quad \textbf{- use inegration by parts}$$

$$u = \ln(x) \quad dv = dx$$

$$du = \frac{1}{x} \qquad v = x$$

$$\int_1^e \ln(x)dx = x\ln x\Big|_1^e - \int_1^e 1dx = e\ln e - 1\ln 1 - \left(x\Big|_1^e\right) = e - e + 1 = 1$$

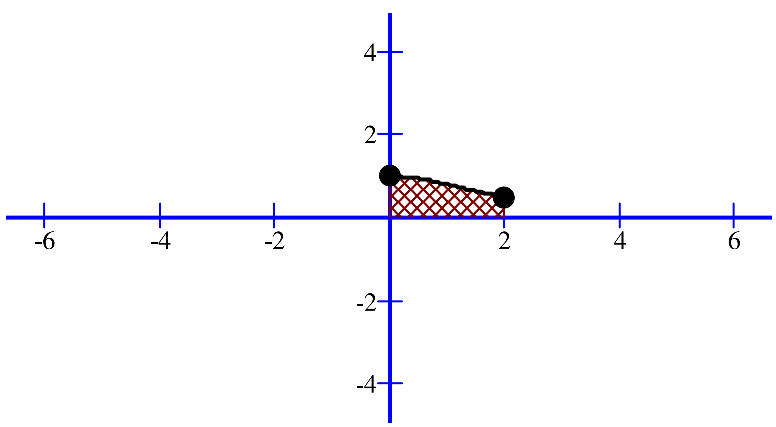


B.

$$\int_0^2 \frac{4}{4+x^2}dx = \frac{4}{4}\int_0^2 \frac{1}{1+\left(\frac{x}{2}\right)^2}dx = 2\int_0^1 \frac{1}{1+u^2}du = 2\arctan u\Big|_0^1 = 2\arctan 1 - 2\arctan 0$$

$$= 2\frac{\pi}{4} - 0$$

$$= \frac{\pi}{2}$$



C.

$$\int \frac{2x+1}{x^2+2x-3}dx = \int \frac{2x+1}{(x+3)(x-1)}dx = \int \left[\frac{A}{x+3} + \frac{B}{x-1}\right]dx$$

$$2x+1 = A(x-1) + B(x+3) = (A+B)x + 3B - A$$

so

$$2 = A + B; 1 = -A + 3B$$

$$\Rightarrow A = \frac{5}{4} \ and \ B = \frac{3}{4}$$

$$\therefore \int \frac{2x+1}{x^2+2x-3}dx = \frac{5}{4}\int \frac{1}{x+3}dx + \frac{3}{4}\int \frac{1}{x-1}dx$$

$$= \frac{5}{4}\ln|x+3| + \frac{3}{4}\ln|x-1| + C$$

D.

$$\int \frac{dx}{\sqrt{9x^2-16}} = \frac{1}{3}\int \frac{dx}{\sqrt{x^2 - \left(\frac{4}{3}\right)^2}} : \qquad x = \frac{4}{3}\sec\theta; \sec\theta = \frac{3x}{4}; dx = \frac{4}{3}\sec\theta\tan\theta d\theta$$

$$\therefore \int \frac{dx}{\sqrt{9x^2-16}} = \frac{1}{3}\int \frac{\sec\theta\tan\theta}{\tan\theta}d\theta = \frac{1}{3}\int \sec\theta d\theta$$

$$= \frac{1}{3}\ln|\sec\theta + \tan\theta| + C$$

$$= \frac{1}{3}\ln\left|\frac{3x}{4} + \frac{\sqrt{9x^2-16}}{4}\right| + C$$

*Note: Maple returns

```
> simplify(int(1/sqrt(9*x^2-16),x));
```

$$\frac{1}{3}\ln(3x + \sqrt{9x^2 - 16})$$

Where did the denominator of 4 go?