Differentiability for Functions of Two Variables

Definition

A function \( f(x, y) \) is **differentiable at point** \( (a, b) \) if there is a linear function \( L(x, y) = f(a, b) + m(x - a) + n(y - b) \) such that if the error \( E(x, y) \) is defined by

\[
E(x, y) = f(x, y) - L(x, y),
\]

and if \( h = x - a, \ k = y - b \), then the relative error satisfies

\[
\lim_{{h \to 0, \ k \to 0}} \frac{E(a + h, b + k)}{\sqrt{h^2 + k^2}} = 0.
\]

The function \( f \) is **differentiable on a region** \( R \) if it is differentiable at each point of \( R \).

- Informally, \( f(x, y) \) is differentiability at \( (a, b) \) if it is “well approximated” by a linear function \( L(x, y) \) near \( (a, b) \).
- If \( f(x, y) \) is differentiable, then partial derivatives must exist.
Partial Derivatives and Differentiability

Fact

If \( f \) is a differentiable function with local linearization
\[
L(x, y) = f(a, b) + m(x - a) + n(y - b),
\]
then \( m = f_x(a, b) \) and \( n = f_y(a, b) \).

Proof: Suppose \( h > 0 \). By definition of differentiability,

\[
0 = \lim_{h \to 0} \frac{E(a + h, b)}{\sqrt{h^2 + 0}} = \lim_{h \to 0} \frac{E(a + h, b)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(a + h, b) - L(a + h, b)}{h} = \lim_{h \to 0} \frac{f(a + h, b) - (f(a, b) + mh)}{h}
\]

\[
= \lim_{h \to 0} \left( \frac{f(a + h, b) - f(a, b)}{h} \right) - m = f_x(a, b) - m.
\]

The same argument holds also for \( h < 0 \). So \( f_x(a, b) = m \). Similarly, \( f_y(a, b) = n \).
Differentiability vs. Existence of Partial Derivatives

Fact

$\text{Differentiability} \Rightarrow \text{Existence of Partial Derivatives.}$

Example

Consider function $f(x, y) = \sqrt{x^2 + y^2}$. Is $f$ differentiable at the origin?

- Related question: Is $f(x) = |x|$ differentiable at $x = 0$?
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Solution 1: Show that partial derivatives do not exist.

Consider

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2 + 0}}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

The limit of $|h|/h$ does not exist as $h \to 0$, as it is $\pm 1$ depending on whether $h > 0$ or $h < 0$. Therefore, it is not differentiable.
Solution 2: Prove by contradiction.
Assume local linearization \( L(x, y) = f(a, b) + m(x - a) + n(y - b) \) exists. Then
\[
E(x, y) = \sqrt{x^2 + y^2} - mx - ny
\]
and
\[
\lim_{h \to 0, k \to 0} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = \lim_{h \to 0, k \to 0} \frac{\sqrt{h^2 + k^2} - mh - nk}{\sqrt{h^2 + k^2}},
\]
Take \( k = 0 \), and then \( \lim_{h \to 0} \frac{|h| - mh}{|h|} = 1 - m \lim_{h \to 0} \frac{h}{|h|} \), for which the limit exists only if \( m = 0 \), but the limit is 1 instead of 0.
Differentiability vs. Existence of Partial Derivatives

**Fact**

*Existence of Partial Derivatives* \( \not\Rightarrow \) *Differentiability.*

**Example**

Consider function \( f(x, y) = \sqrt[3]{xy} \). Show that its partial derivatives exist but \( f \) is not differentiable at \((0, 0)\).

**Answer:**

\[
 f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0.
\]

Similarly \( f_y(0, 0) = 0 \). If \( f(x, y) \) is differentiable, its local linearization would be \( L(x, y) = f(0, 0) = 0 \) and \( E(x, y) = f(x, y) \). However, the limit of \( E(x, y)/\sqrt{h^2 + k^2} \) does not exist. Specifically, consider direction of \( h = k \).

\[
 \lim_{h \to 0, k \to 0} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = \lim_{h \to 0} \frac{3\sqrt{h^2}}{\sqrt{h^2 + h^2}} = \lim_{h \to 0} \frac{1}{h^{1/3}\sqrt{2}} = \infty.
\]
Differentiability vs. Continuity

- Differentiability implies continuity, but not vice versa
- Does existence of partial derivatives imply continuity?
- Example: Consider function

\[ f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases} \]

- Partial derivatives are

\[ f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0 - 0}{h} = 0. \]

Similarly \( f_y(0, 0) = 0 \). However, \( f(x, y) \) is not continuous at \( (0, 0) \) as along the direction of \( x = y \), \( f(x, y) = 1/2 \) instead of 0.
Observations

1. If a function is differentiable at a point, then both partial derivatives exist there.

2. Having both partial derivatives at a point does not guarantee that a function is differentiable there.

3. If a function is differentiable at a point, then it is continuous there.

4. Having both partial derivatives at a point does not guarantee that a function is continuous there.
Continuity of Partial Derivatives Implies Differentiability

**Theorem**

*If the partial derivatives, \( f_x \) and \( f_y \), of a function \( f \) exist and are continuous on a small disk centered at the point \((a, b)\), then \( f \) is differentiable at \((a, b)\).*

- **Note:** In effect, the requirement of continuous partial derivatives is more stringent than that of differentiability.
- There exist some differentiable functions which do not have continuous partial derivatives.
- Functions with continuous partial derivatives are called \( C^1 \).
Show that function \( f(x, y) = \ln(x^2 + y^2) \) is differentiable everywhere in its domain.

Answer:

- The domain of \( \ln(x^2 + y^2) \) is everywhere except for the origin.
- The partial derivatives are

\[
    f_x = \frac{2x}{x^2 + y^2} \quad \text{and} \quad f_y = \frac{2y}{x^2 + y^2}.
\]

- Numerator and denominators are continuous everywhere except at origin, so are \( f_x \) and \( f_y \). Thus, \( f \) is differentiable everywhere in its domain.
Summary

- Four related levels of continuities
  - Continuity at point
  - Existence of partial derivatives
  - Differentiability at point
  - Continuity of partial derivatives

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