AMS 261: Applied Calculus III (Multivariable Calculus)

Lecture 16: Differentiability

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Differentiability for Functions of Two Variables

Definition

A function $f(x,y)$ is ***differentiable at point*** (a,b) if there is a linear function $L(x,y) = f(a,b) + m(x-a) + n(y-b)$ such that if the error $E(x,y)$ is defined by

$$E(x,y) = f(x,y) - L(x,y),$$

and if $h = x - a$, $k = y - b$, then the relative error satisfies

$$\lim_{h\to 0, k\to 0} \frac{E(a+h, b+k)}{\sqrt{h^2+k^2}} = 0.$$

The function f is ***differentiable on a region*** R if it is differentiable at each point of R.

- Informally, $f(x,y)$ is differentiability at (a,b) if it is "well approximated" by a linear function $L(x,y)$ near (a,b)
- If $f(x,y)$ is differentiable, then partial derivatives must exist.

Partial Derivatives and Differentiability

Fact

If f is a differentiable function with local linearization $L(x, y) = f(a, b) + m(x - a) + n(y - b)$, then $m = f_x(a, b)$ and $n = f_y(a, b)$.

Proof: Suppose $h > 0$. By definition of differentiability,

$$
\begin{aligned}
0 &= \lim_{h\to 0} \frac{E(a+h, b)}{\sqrt{h^2+0}} = \lim_{h\to 0} \frac{E(a+h, b)}{h} \\
&= \lim_{h\to 0} \frac{f(a+h, b) - L(a+h, b)}{h} = \lim_{h\to 0} \frac{f(a+h, b) - (f(a, b) + mh)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(a+h, b) - f(a, b)}{h} \right) - m = f_x(a, b) - m.
\end{aligned}
$$

The same argument holds also for $h < 0$. So $f_x(a, b) = m$. Similarly, $f_y(a, b) = n$.

Differentiability vs. Existence of Partial Derivatives

Fact

Differentiability $\Rightarrow$ Existence of Partial Derivatives.

Example

Consider function $f(x, y) = \sqrt{x^2 + y^2}$. Is f differentiable at the origin?

- Related question: Is $f(x) = |x|$ differentiable at $x = 0$?

Differentiability vs. Existence of Partial Derivatives

Fact

Differentiability ⇏ Existence of Partial Derivatives.

Example

Consider function $f(x, y) = \sqrt{x^2 + y^2}$. Is f differentiable at the origin?

- Related question: Is $f(x) = |x|$ differentiable at $x = 0$?

Answer: No (for both).

Differentiability vs. Existence of Partial Derivatives

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Differentiability $\Rightarrow$ *Existence of Partial Derivatives.*

Example

Consider function $f(x,y) = \sqrt{x^2 + y^2}$. Is f differentiable at the origin?

- Related question: Is $f(x) = |x|$ differentiable at $x = 0$?

Answer: No (for both).
Solution 1: Show that partial derivatives do not exist.
Consider

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{\sqrt{h^2 + 0}}{h} = \lim_{h \to 0} \frac{|h|}{h}.$$

The limit of $|h|/h$ does not exist as $h \to 0$, as it is ± 1 depending on whether $h > 0$ or $h < 0$. Therefore, it is not differentiable.

Example Cont'd

Solution 2: Prove by contradiction.
Assume local linearization $L(x, y) = f(a, b) + m(x - a) + n(y - b)$ exists. Then

$$E(x, y) = \sqrt{x^2 + y^2} - mx - ny$$

and

$$\lim_{h\to 0, k\to 0} \frac{E(h, k)}{\sqrt{h^2 + k^2}} = \lim_{h\to 0, k\to 0} \frac{\sqrt{h^2 + k^2} - mh - nk}{\sqrt{h^2 + k^2}},$$

Take $k = 0$, and then $\lim_{h\to 0} \frac{|h| - mh}{|h|} = 1 - m \lim_{h\to 0} \frac{h}{|h|}$, for which the limit exists only if $m = 0$, but the limit is 1 instead of 0.

Differentiability vs. Existence of Partial Derivatives

Fact

Existence of Partial Derivatives $\not\Rightarrow$ Differentiability.

Example

Consider function $f(x,y) = \sqrt[3]{xy}$. Show that its partial derivatives exist but f is not differentiable at $(0,0)$.

- Answer:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly $f_y(0,0) = 0$. If $f(x,y)$ is differentiable, its local linearization would be $L(x,y) = f(0,0) = 0$ and $E(x,y) = f(x,y)$. However, the limit of $E(x,y)/\sqrt{h^2+k^2}$ does not exist. Specifically, consider direction of $h = k$.

$$\lim_{h\to 0, k\to 0} \frac{E(h,k)}{\sqrt{h^2+k^2}} = \lim_{h\to 0} \frac{\sqrt[3]{h^2}}{\sqrt{h^2+h^2}} = \lim_{h\to 0} \frac{1}{h^{1/3}\sqrt{2}} = \infty.$$

Differentiability vs. Continuity

- Differentiability implies continuity, but not vice versa
- Does existence of partial derivatives imply continuity?
- Example: Consider function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

- Partial derivatives are

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly $f_y(0,0) = 0$. However, $f(x,y)$ is not continuous at $(0,0)$ as along the direction of $x = y$, $f(x,y) = 1/2$ instead of 0.

Observations

1. If a function is differentiable at a point, then both partial derivatives exist there
2. Having both partial derivatives at a point does not guarantee that a function is differentiable there
3. If a function is differentiable at a point, then it is continuous there.
4. Having both partial derivatives at a point does not guarantee that a function is continuous there.

Continuity of Partial Derivatives Implies Differentiability

Theorem

If the partial derivatives, f_x and f_y, of a function f exist and are continuous on a small disk centered at the point (a, b), then f is differentiable at (a, b)
.

- Note: In effect, the requirement of continuous partial derivatives is more stringent than that of differentiability
- There exist some differentiable functions which do not have continuous partial derivatives
- Functions with continuous partial derivatives are called C^1

Example

Example

Show that function $f(x, y) = \ln(x^2 + y^2)$ is differentiable everywhere in its domain.

Answer:

- The domain of $\ln(x^2 + y^2)$ is everywhere except for the origin.
- The partial derivatives are

$$f_x = \frac{2x}{x^2 + y^2} \text{ and } f_y = \frac{2y}{x^2 + y^2}.$$

- Numerator and denominators are continuous everywhere except at origin, so are f_x and f_y. Thus, f is differentiable everywhere in its domain.

Summary

- Four related levels of continuities
  - Continuity at point
  - Existence of partial derivatives
  - Differentiability at point
  - Continuity of partial derivatives

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